THREE ERAS OF PYTHAGOREANISM

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WARNING!

Do not believe anything I say!

I am a musician, not a philosopher!

Only a fool would trust a musician about philosophy!

… on the other hand …

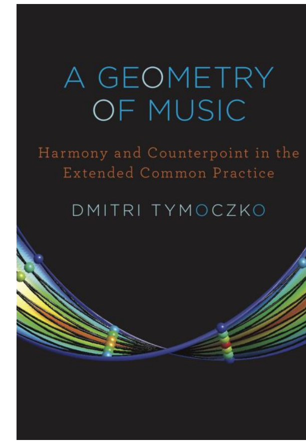
I was raised by a philosopher

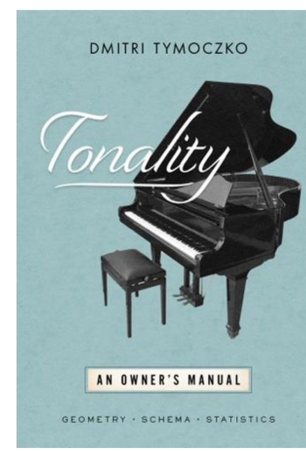
I am married to a philosopher

I was kicked out of philosophy grad. school by a philosopher.

… on the other hand …

When it comes to the mathematics of music I have some claim to knowledge …





THE ERAS

Metaphysical Pythagoreanism (~500 BCE–)

human beings are directly sensitive to number

the world is structured by simple ratios and shapes

Enlightenment Pythagoreanism (~1700–)

our relation to number is mediated by continuous spaces

the world is structured by differential equations

Categorical Pythagoreanism (~1945–)

continuity is inessential

deeper and more abstract patterns of reasoning crossing conceptual boundaries

I am going to illustrate each era with reference to the consonance of the perfect fifth.

FIRST ERA

PYTHAGORAS (

HIPPASUS?

- Heard some blacksmiths pounding hammers?
- Conducted experiments with bronze disks?
- Reinvented the monochord?

… discovered that consonance was connected to whole-number ratios …

"the first empirically secure mathematical description of a physical fact" (Wikipedia)

METAPHYSICAL PYTHAGOREANISM

- No theory of how whole numbers produce the sensation of consonance.
  - A direct or unmediated sensitivity to numerical proportion
  - Led to a metaphysical picture emphasizing numerical simplicity
    - *musica universalis*
    - circular planetary orbits
    - the theory of forms?

(VERY OBVIOUS) PROBLEMS

- A simple ratio is a linear relationship with small integer coefficients

  $i\mathbf{x} = j\mathbf{y}$ *for small integers i, j*
- Such relationships are just a small part of science and math.
- There are lots of relationships with
  - noninteger coefficients ($\boldsymbol{c} = \pi \boldsymbol{d}$)
  - exponents ($\boldsymbol{a} = \pi \boldsymbol{r}^2$)
- To be sure, many musicians still seem to subscribe to metaphysical Pythagoreanism:
  - many seem to think there is something special about small-number ratios like 7:4

SECOND ERA

ENLIGHTENMENT PYTHAGOREANISM

- Our interaction with numbers is mediated by continuous spaces and differential equations.
- It is the equations that are mathematically simple rather than the numerical quantities in themselves.
- The biggie:

$$d = vt + \frac{1}{2}at^2$$

ENLIGHTENMENT PYTHAGOREANISM

- Our interaction with numbers is mediated by continuous spaces and differential equations.
- It is the equations that are mathematically simple rather than the numerical quantities in themselves.
- Kepler's "harmonic law."

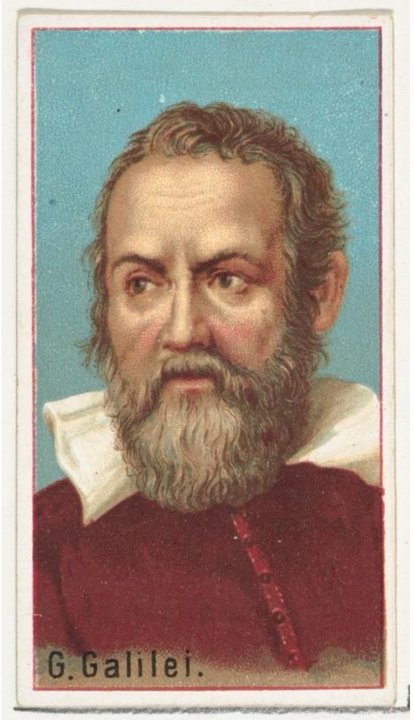
$$T \propto r^{3/2}$$ *same ratio as perfect fifth!*

LAGRANGE/HAMILTON

- The state of the entire world (or any isolated subsystem of the world) can be represented as a point in a configuration space.
- That point moves smoothly along a trajectory determined by differential equations whose general form is fixed.
- Physical theory is fundamentally continuous

SOME HISTORY

- Enlightenment Pythagoreanism:
  - "The book of nature is written in the language of mathematics."
  - Galileo Galilei (1520–1591)
  - *Equations* not *numbers*.
  - *Solution spaces* not points.
- Less known:
  - Galileo's father Vincenzo was a famous composer/theorist.
  - The idea of mathematizing nature descends through him from Pythagoras!



INTERESTING QUESTION

- Why are physical equations *simple*?
- That is, why do they tend to involve small exponents?
  - arise from *our* approximations
  - stability theorems in dynamical systems
  - we can't quantize equations with more than two time derivatives
  - …
- Unlike metaphyiscal Pythagoreanism, we have some answers.

ENLIGHTENMENT PYTHAGOREANISM: TWO CASE STUDIES

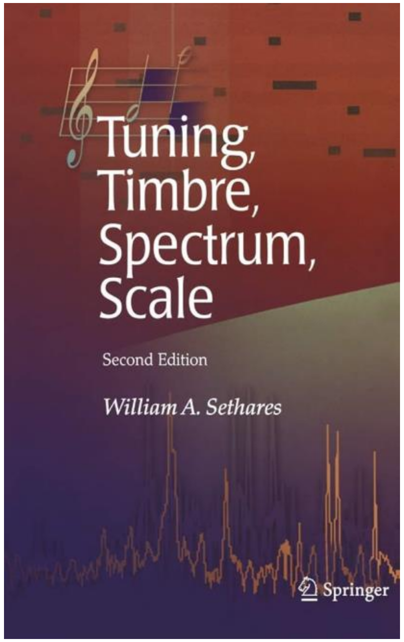
1. Where does consonance come from?
2. How is musical knowledge geometrical?

CASE STUDY 1

- Consonance is *not* produced by numbers themselves.
- Instead, it arises through a process of *dissonance minimization* that depends on the structure of a sound, which in turn depends on the physics of the instrument involved.
- It is a *minimum* in a continuous space.

FOURIER/HELMHOLTZ/SETHARES

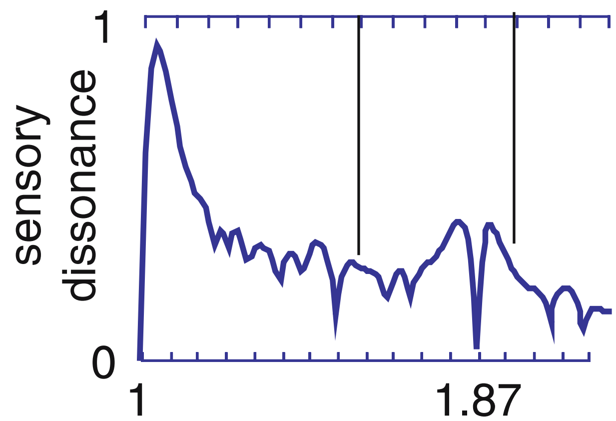
- A vibrating body produces *partials* according to its detailed physical construction.
- Consonance is produced when the partials of two sounds coincide (or are sufficiently distant so as not to interfere).
- Familiar instruments vibrate harmonically:
  - f, $2f$, $3f$, $4f$, …
- The consonance of whole-number frequency ratios is due to the *spectrum* of the sound.



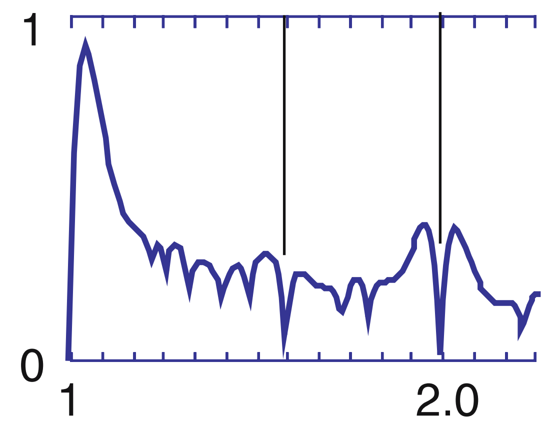
FOURIER/HELMHOLTZ/SETHARES

- Therefore, nonharmonic sounds would sound consonant at different, non-whole-number ratios.
  - familiar scale, harmonic partials
  - familiar scale, partials stretched by a factor of 2.1
  - stretched scale, stretched partials
  - stretched scale, familiar partials

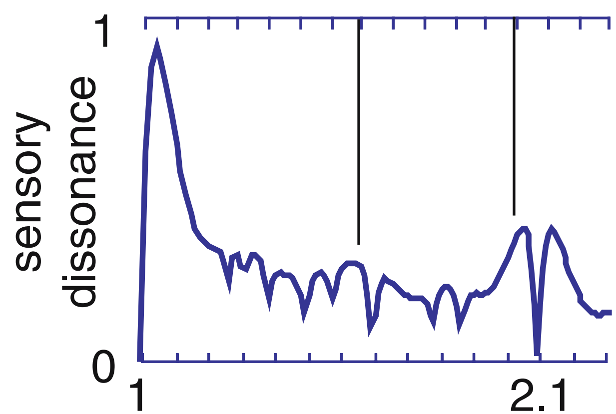
DISSONANCE CURVES (SETHARES)



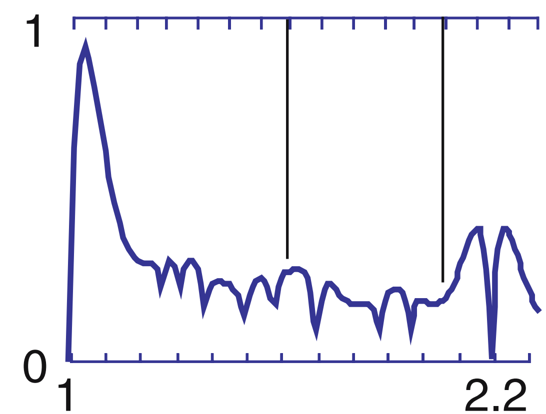
1.87 compressed spectrum and scale



2.0 harmonic spectrum and scale

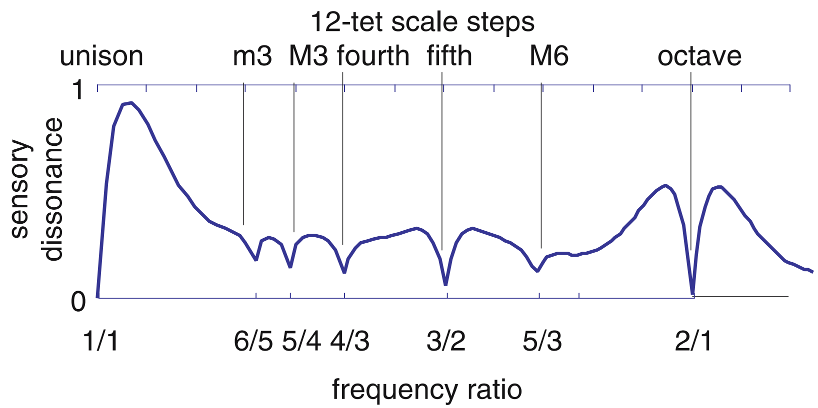


2.1 stretched spectrum and scale

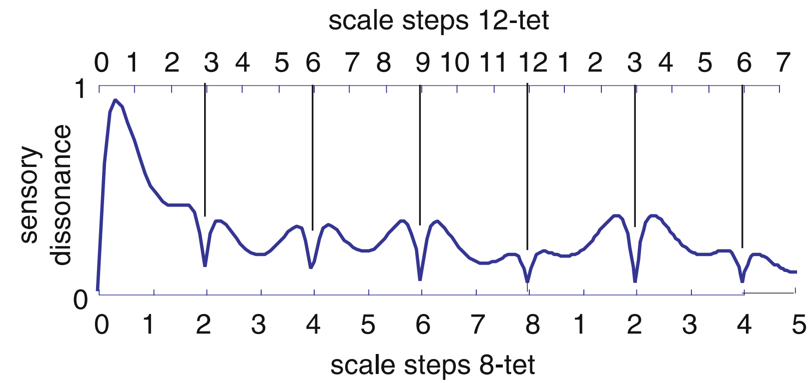


2.2 stretched spectrum and scale

DISSONANCE CURVES (SETHARES)



harmonic spectrum

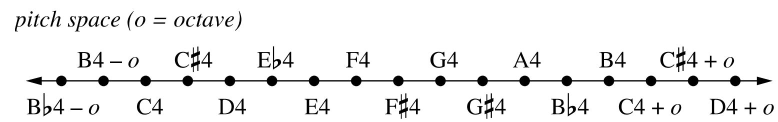


8tet spectrum

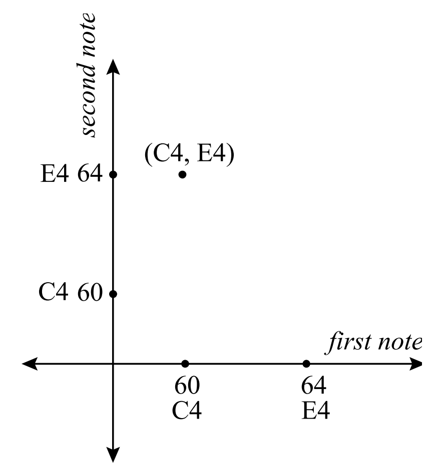
so …

- Pythagoras didn't see the whole picture.
- Simple ratios are a byproduct:
  - consonant ratios are determined by a sound's *spectrum* (overtone structure)
  - many Western instruments produce harmonic spectra, leading to simple-ratio consonance
  - other instruments (bells, metallophones) produce nonharmonic spectra
  - these instruments do not minimize dissonance at simple ratios.

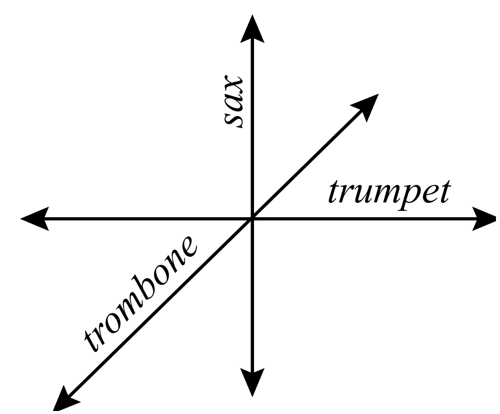
CASE STUDY 2



- Pitch is a continuous one-dimensional space.



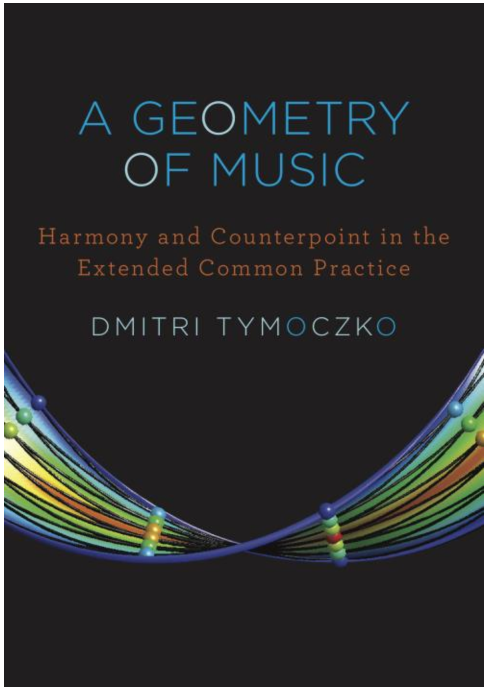
- An ensemble of instruments occupies a point in a higher-dimensional *configuration space*.
  - The product of many copies of pitch space.



- Symmetries allow us to fold up Euclidean space, creating a variety of *quotient spaces* with interesting geometrical features.

SYMMETRY AND GEOMETRY

- The **O**ctave symmetry turns lines into circles
- The **P**ermutation symmetry adds mirrors
- The **T**ransposition symmetry subtracts a dimension
- The **I**nversion symmetry adds mirrors
- *Loops* in these quotient spaces ("orbifolds") represent musical transformations connecting an object to itself
  - "dual" to symmetries in a way to be described



EXAMPLE: OCTAVE SHIFTS

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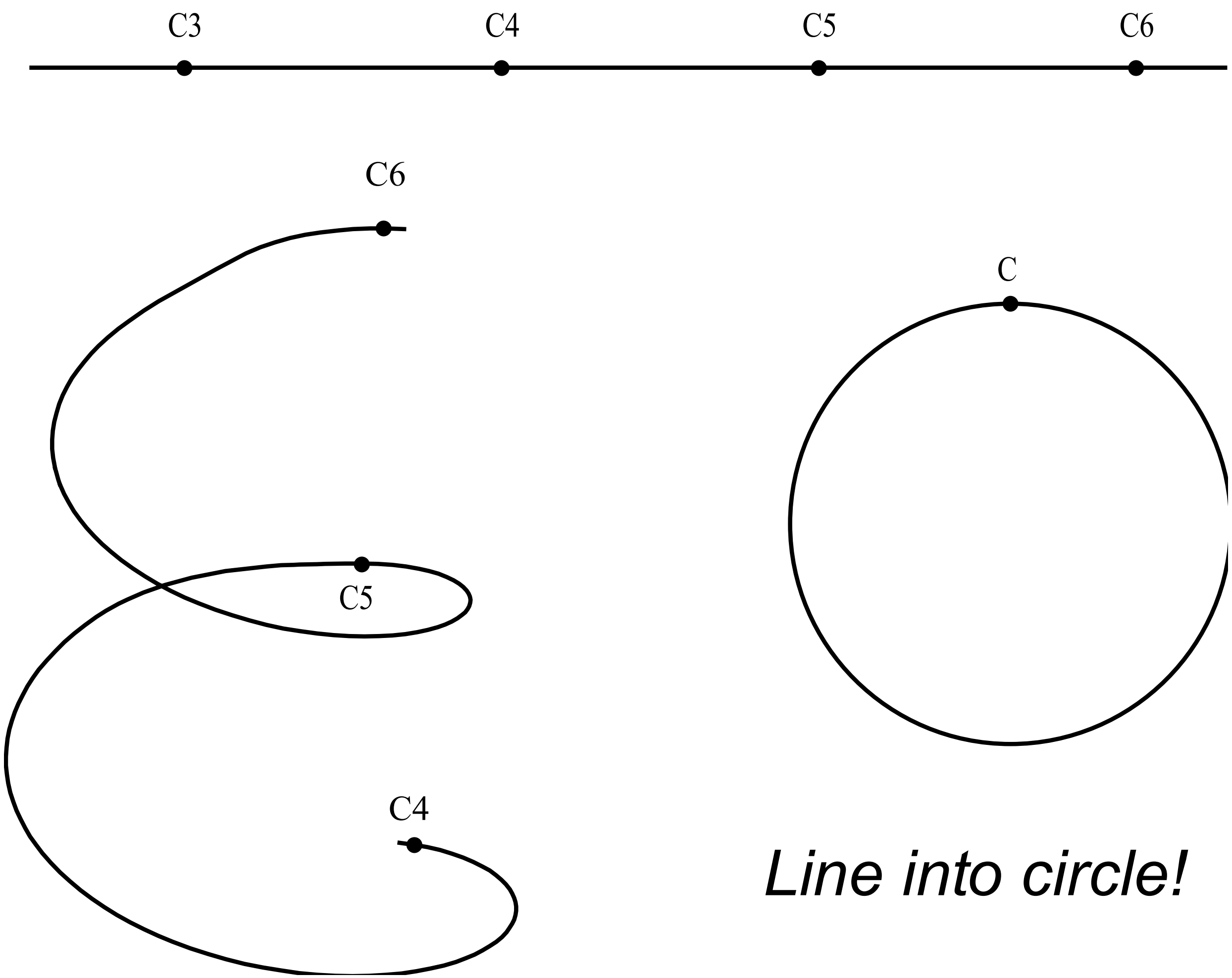
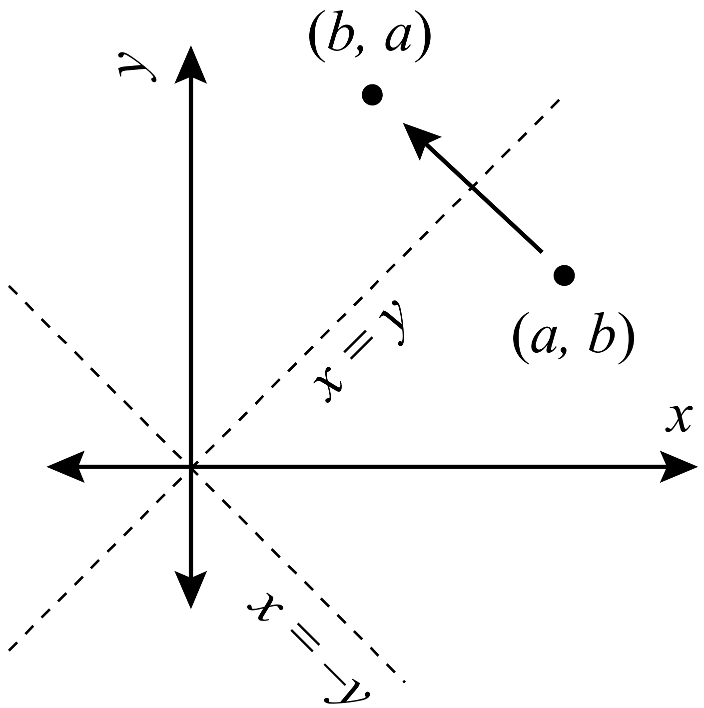
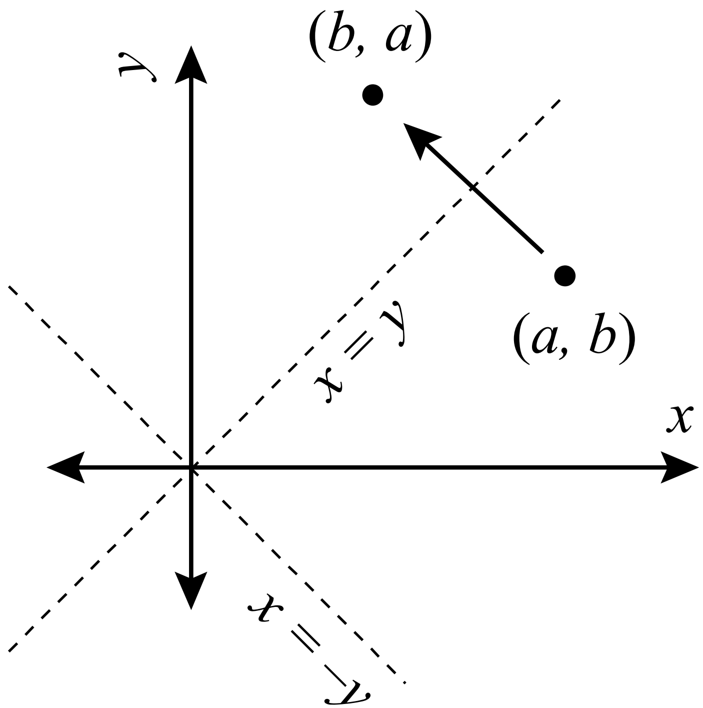


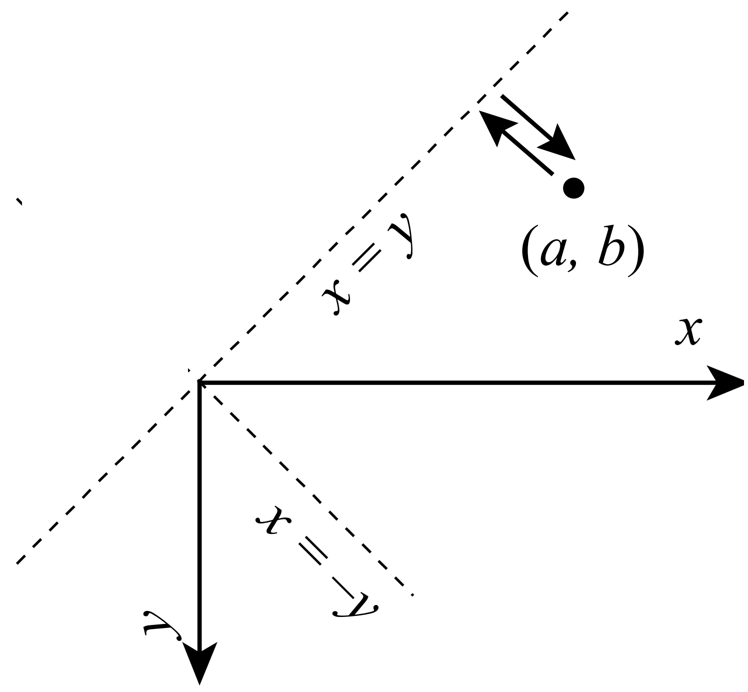
Line into circle!

EXAMPLE: PERMUTATION



EXAMPLE: PERMUTATION





Add mirror!

EXAMPLE: OCTAVE SHIFTS AND PERMUTATION

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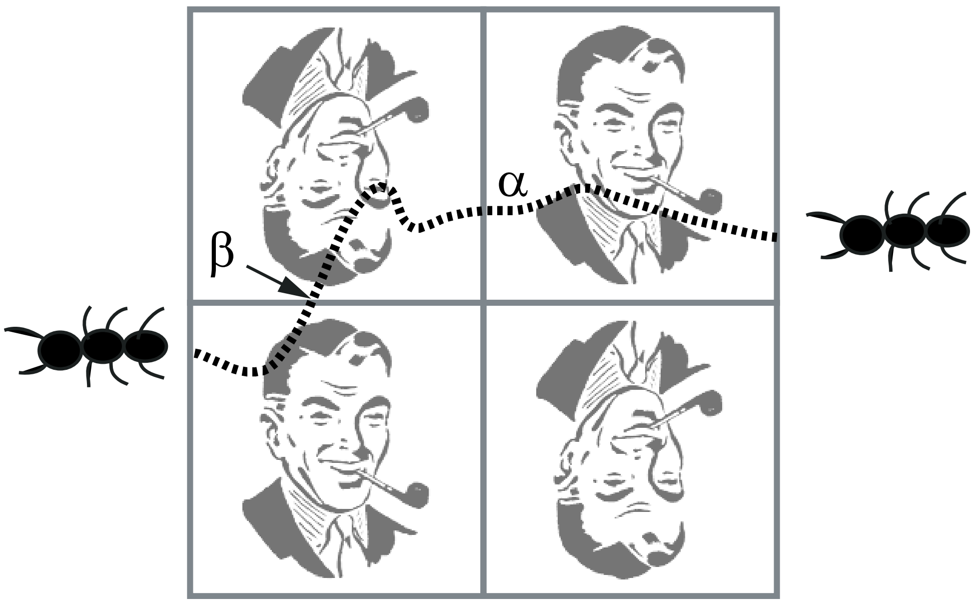
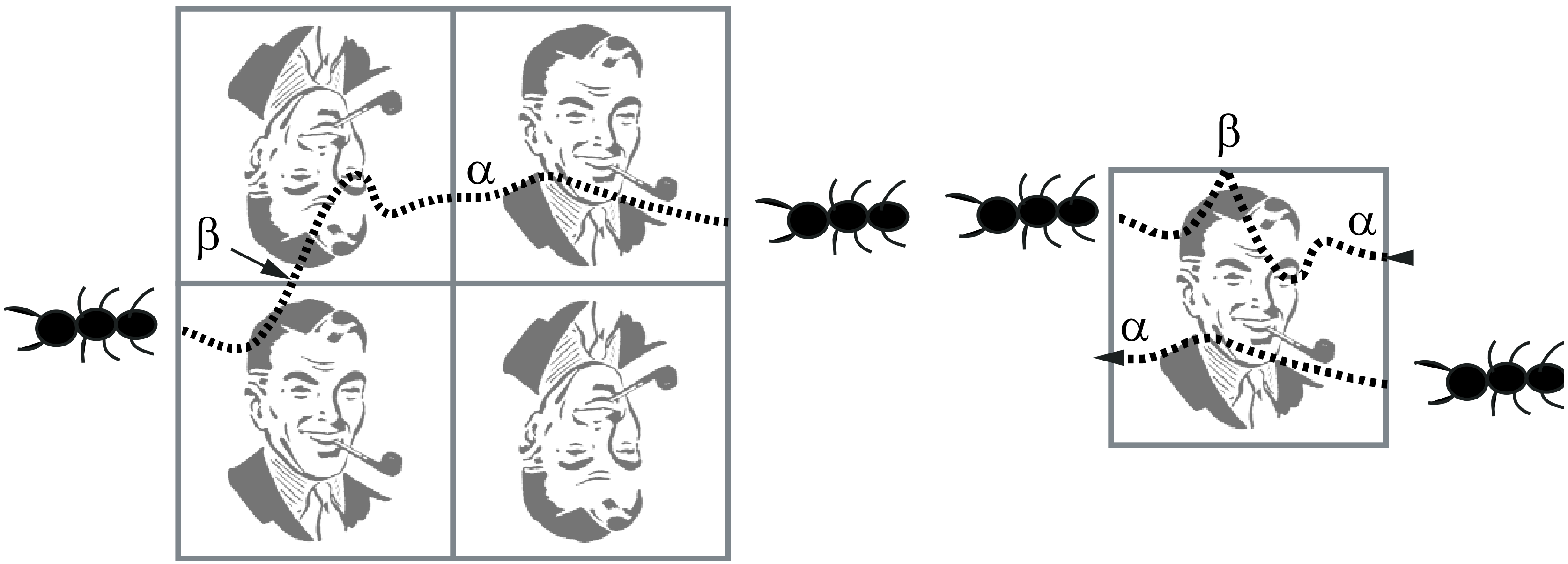


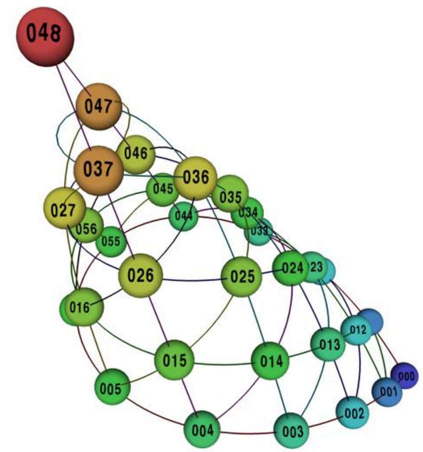
EXAMPLE: OCTAVE SHIFTS AND PERMUTATION



Circle (α) and mirror (β)!

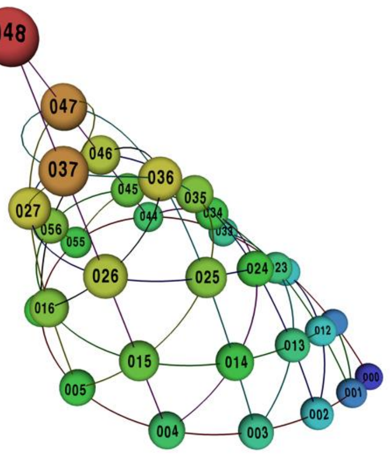
MUSICAL KNOWLEDGE AS GEOMETRICAL KNOWLEDGE

- Musicians learn their way around complicated, continuous geometrical quotient spaces.
- Often they exploit short-distance loops in these spaces.
- The structure of these spaces helps explain many features of Western music.



MUSICAL KNOWLEDGE AS GEOMETRICAL KNOWLEDGE

- By exploring the structure of these spaces we can uncover the conditions of possibility of certain types of musical organization
  - in particular: combining harmony and melody.
- Musical knowledge is knowledge of a continuous space of possibilities.



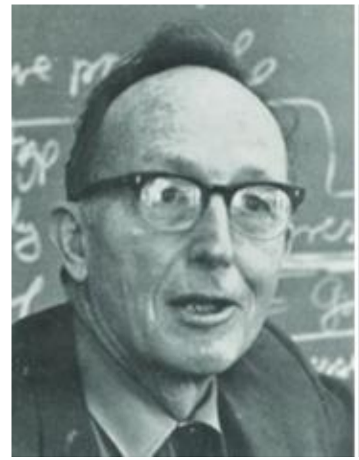
CONFESSION

- For many years I believed continuity was **absolutely essential** to these models.
  - I could be quite forceful in advocating for the importance of continuity in musical modeling.
- This was wrong.
  - Continuity is **completely irrelevant.**
- There is a pretty big gap between "absolutely essential" and "completely irrelevant"!

THIRD ERA

CATEGORY THEORY

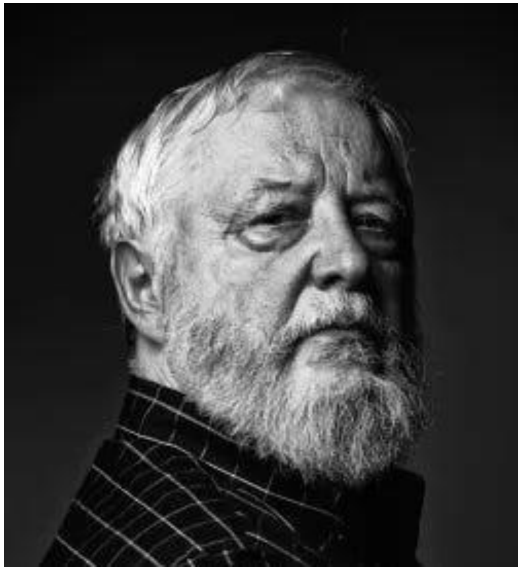
- Invented by Eilenberg and Mac Lane in 1945.
- Motivated by the desire to describe connections across mathematical areas
  - e.g. between the continuous world of topology and the discrete world of number theory.


Mac Lane


Eilenberg

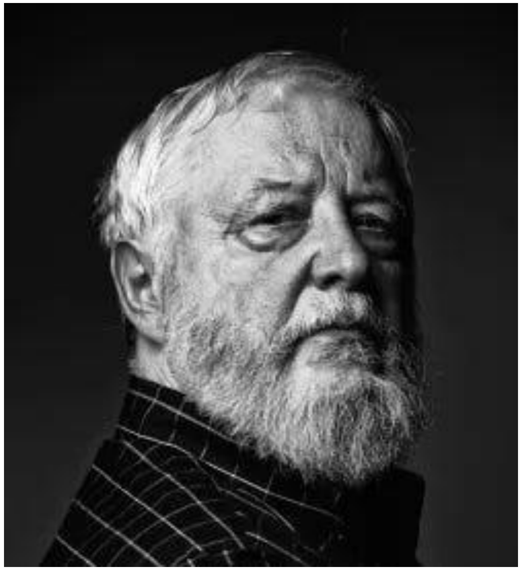
CATEGORY THEORY

- Developed by Alexander Grothendieck
  - used it to radically transform the foundations of algebraic geometry
- William Lawvere connected category theory to traditional themes of philosophy.



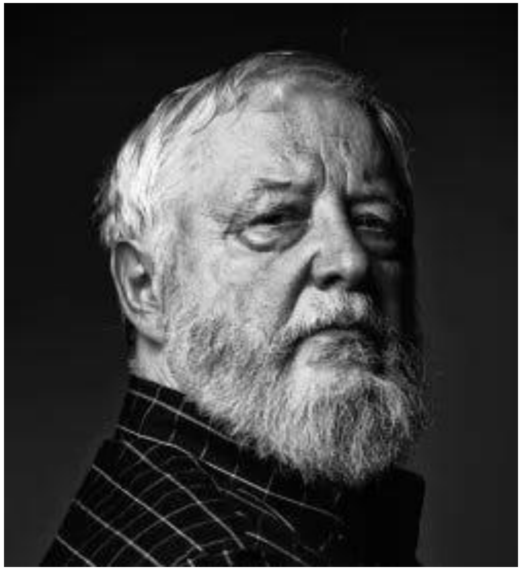
LAWVERE

The technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, ***objective vs. subjective,*** *being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc.*



LAWVERE (2)

Here by "rig" we mean a structure like a commutative ring except that it need not have negatives, and the name of Burnside was suggested by Dress to denote the process of abstraction [...] ***which Cantor learned from Steiner****: the isomorphism classes of objects from a given distributive category form a rig when multiplied and added using product and coproduct; the algebra of this Burnside rig partly reflects the properties of the category and also partly measures the spaces in it in a way which (as suggested by Mayberry)* ***gives deeper significance to the statement attributed to Pythagoras: "Each thing is number."***



CATEGORY THEORY: GREATEST HITS

- From 1940–1960 mathematicians noticed analogues to topological phenomena, arising in the *discrete* context of number theory.
- **Q:** what "space" is formed by the integers?
  - what are its points?
- Grothendieck:
  - the integers are a *scheme*
  - their "points" are the *prime numbers!*
- This is a radical rethinking of the foundational concept of "space."

CATEGORY THEORY: GREATEST HITS

- Set theory and logic can be reformulated in the language of category theory.
- Many categories have their own intrinsic logic.
  - the category of sets (**Set**) has a classical logic
  - the category of topological spaces (**Top**) has an intuitionistic logic!
- *Proof* can be modeled as a kind of motion through a space.
- Logic is fundamentally *topological!*

PERSONAL OPINION

- As far as I can tell, category theory seems relevant to a wide range of philosophical questions.
- Philosophers seem to lag behind in their knowledge of the subject.
  - still often think in old-fashioned (set-theoretical) terms

THE BIG PICTURE

- A lot of phenomena we know from topology (and more generally, the continuous world) have discrete analogues.
- Category theory allows us to extract the deep logic from its original, continuous context.
- Topology is more universal than you think.
- Somewhat surprisingly, this is relevant to music theory.

WARM UP EXAMPLE

- Suppose we start with a Pythagorean space of acoustically pure octaves and fifths.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭–1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

WARM UP EXAMPLE

- This is an infinite 2D plane.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭–1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

WARM UP EXAMPLE

- “Pythagorean pitch space”

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭–1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

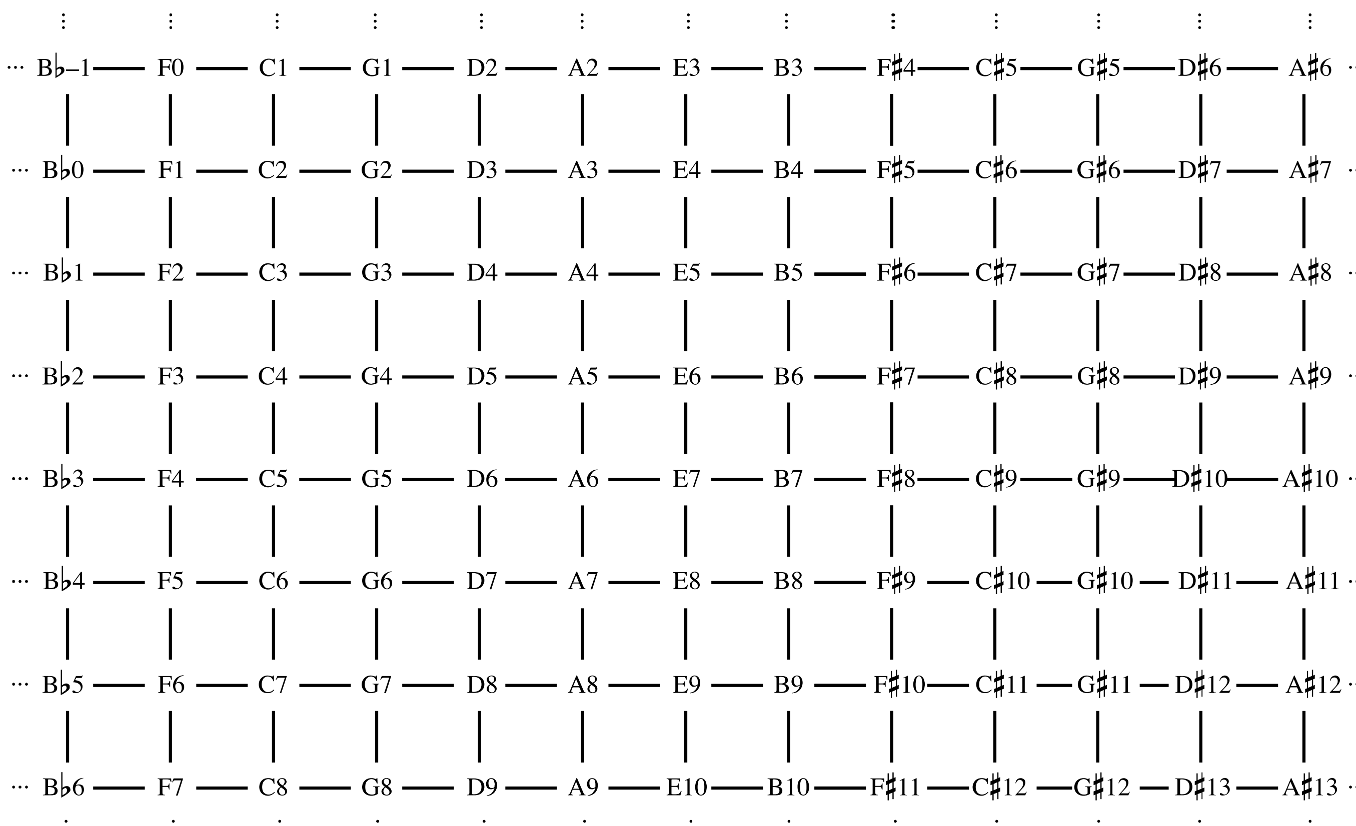
WARM UP EXAMPLE

- It contains an infinite number of points in every octave.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭–1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

WARM UP EXAMPLE

- You cannot make a (linear) keyboard out of this space (with finite keys/octave)!



WARM UP EXAMPLE

- We can make a keyboard capable of playing *all** of these notes by forming a *quotient space*.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭−1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

** in a given octave range*

WARM UP EXAMPLE

- We declare that these two notes, which sound very similar, correspond to the same key.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭−1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

WARM UP EXAMPLE

- We interpret pairs of keys as picking out the *shortest** horizontal path in the space.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
⋯	B♭–1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	⋯
⋯	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	⋯
⋯	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	⋯
⋯	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	⋯
⋯	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	⋯
⋯	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	⋯
⋯	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	⋯
⋯	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

** defaulting to rightward motion in the case of ties*

WARM UP EXAMPLE

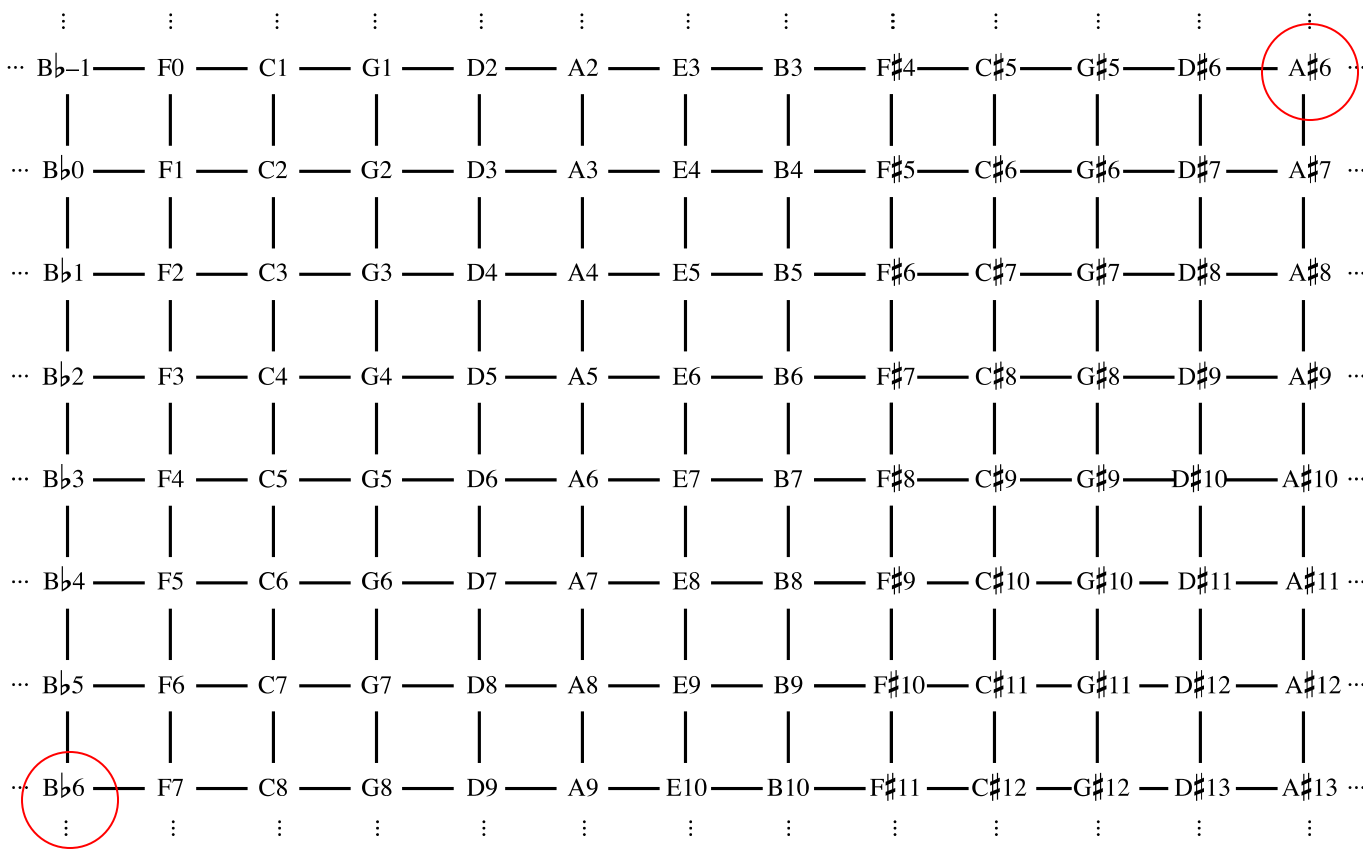
- For the purposes of illustration we can also glue together octave-related pitches.

	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
···	B♭–1	F0	C1	G1	D2	A2	E3	B3	F♯4	C♯5	G♯5	D♯6	A♯6	···
···	B♭0	F1	C2	G2	D3	A3	E4	B4	F♯5	C♯6	G♯6	D♯7	A♯7	···
···	B♭1	F2	C3	G3	D4	A4	E5	B5	F♯6	C♯7	G♯7	D♯8	A♯8	···
···	B♭2	F3	C4	G4	D5	A5	E6	B6	F♯7	C♯8	G♯8	D♯9	A♯9	···
···	B♭3	F4	C5	G5	D6	A6	E7	B7	F♯8	C♯9	G♯9	D♯10	A♯10	···
···	B♭4	F5	C6	G6	D7	A7	E8	B8	F♯9	C♯10	G♯10	D♯11	A♯11	···
···	B♭5	F6	C7	G7	D8	A8	E9	B9	F♯10	C♯11	G♯11	D♯12	A♯12	···
···	B♭6	F7	C8	G8	D9	A9	E10	B10	F♯11	C♯12	G♯12	D♯13	A♯13	···
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

** defaulting to rightward motion in the case of ties*

WARM UP EXAMPLE

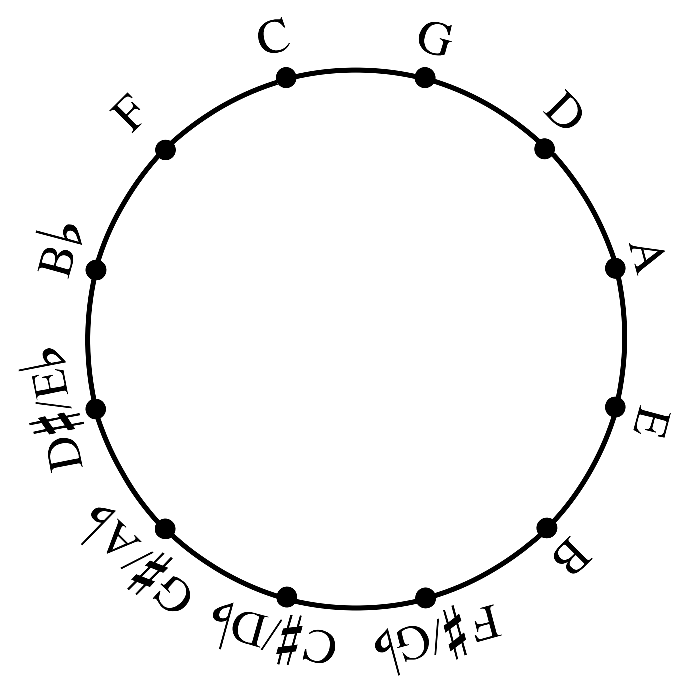
- Simplifies the geometry to one dimension.



** defaulting to rightward motion in the case of ties*

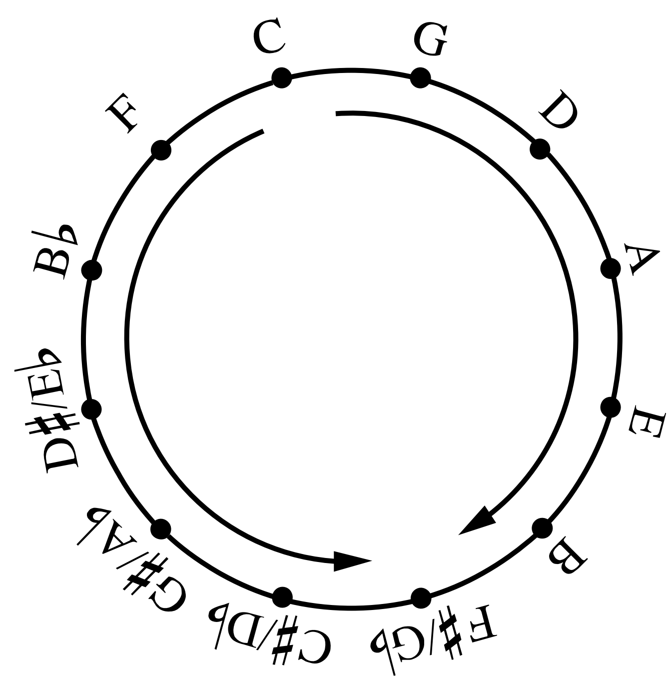
PYTHAGOREAN PITCH-CLASS SPACE

- What results is a quotient space
  - structurally analogous to those of voice-leading geometry.
  - but fundamentally discrete
- Loops in this space represent *comma shifts* rather than octave shifts.
- *Pythagorean pitch-class space.*



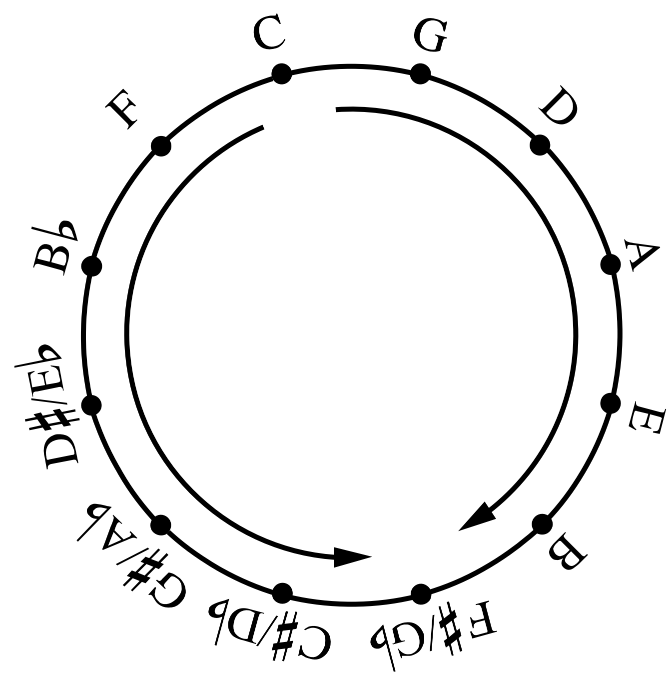
PYTHAGOREAN PITCH-CLASS SPACE

- This is an example of a category-theoretical discovery:
  - familiar topological concepts (quotient space) have discrete analogues
- Musicians inhabit this space!
  - singers, string players



PYTHAGOREAN PITCH-CLASS SPACE

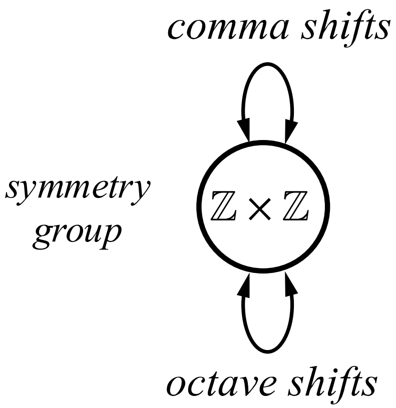
- Category theory models a fundamental mechanism of *concept formation:*
  - the construction of 12 finite pitch-categories (“pitch classes”) from the infinite 2D space of Pythagorean pitches.
- Prerequisite to
  - conventional notation
  - the construction of conventional keyboard instruments



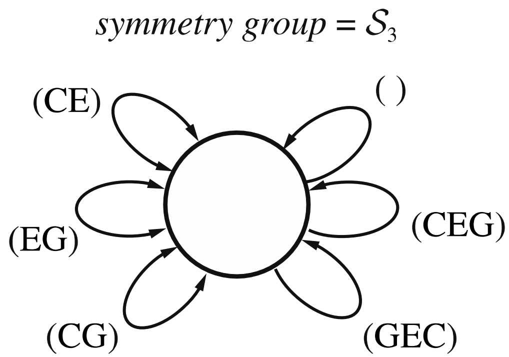
GOAL

- We want to generalize this example.
- We want a *universal theory of symmetry.*
  - applicable whenever any symmetry group acts on any set or space
  - giving us access to familiar toplogical (or *logical*) concepts like quotient space, covering space, homotopy group, etc.
- This can be considered a *topological model of concept formation.*
  - possibly relevant to understanding AI

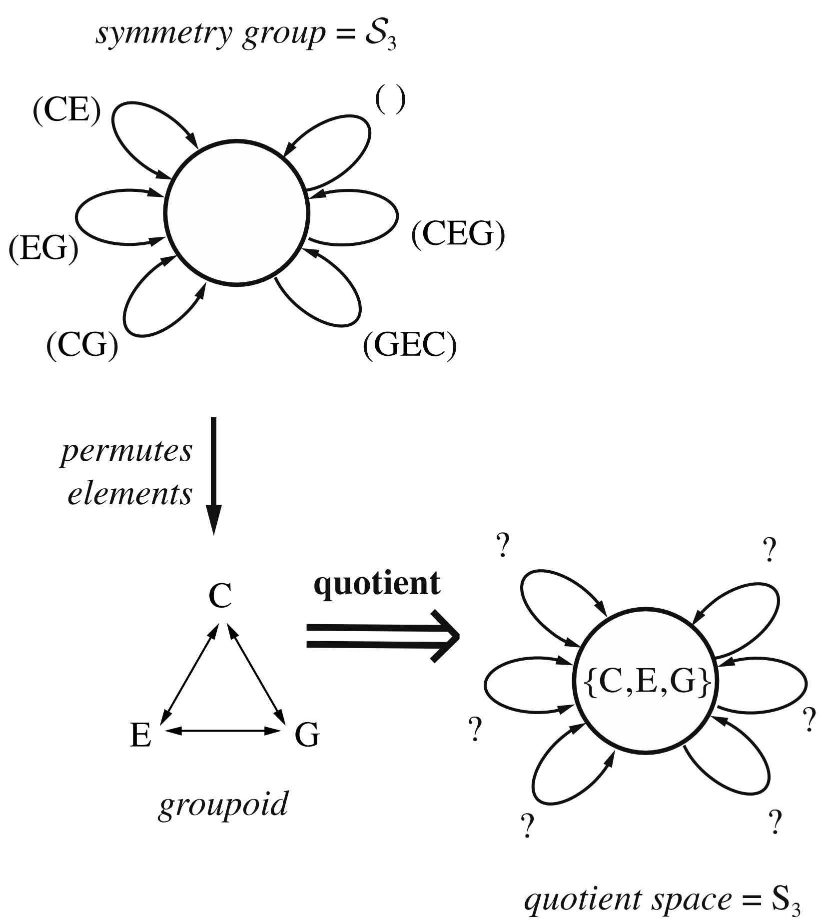
IN A PICTURE


comma shifts
symmetry group
$\mathbb{Z} \times \mathbb{Z}$
octave shifts

OBSTACLE



OBSTACLE


symmetry group = $\mathcal{S}_3$
(CE)
( )
(EG)
(CEG)
(CG)
(GEC)
permutes elements
C
E
G
groupoid
quotient
?
?
{C,E,G}
?
?
?
?
quotient space = S_3

PROBLEM

Standard mathematical techniques do not allow us to interpret the meaning of paths in a generalized quotient space, at least not in a way that is useful for musicians or philosophers.

METAPHYSICS OF PERMUTATION

- Suppose I play the three notes C4, E4, G4 using sine waves at the same volume centered in the stereo field.
- Now I play those same notes again.
- **Q:** did I permute them?
- **A:** huh?

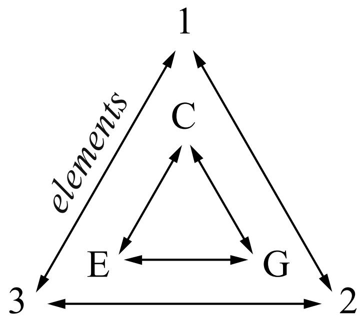
THE PERMUTATION PRINCIPLE

- **PP:** for a permutation to be conceivable, we need multiple ways of identifying objects:
  - content *plus* some additional attribute
  - spatial position, instrumental timbre, loudness, temporal order, octave
- In other words, for a permutation to be conceivable we need both
  - *elements* (“content,” notes)
  - *attributes* (“data,” loudness, timbre, etc.)

THE PERMUTATION PRINCIPLE

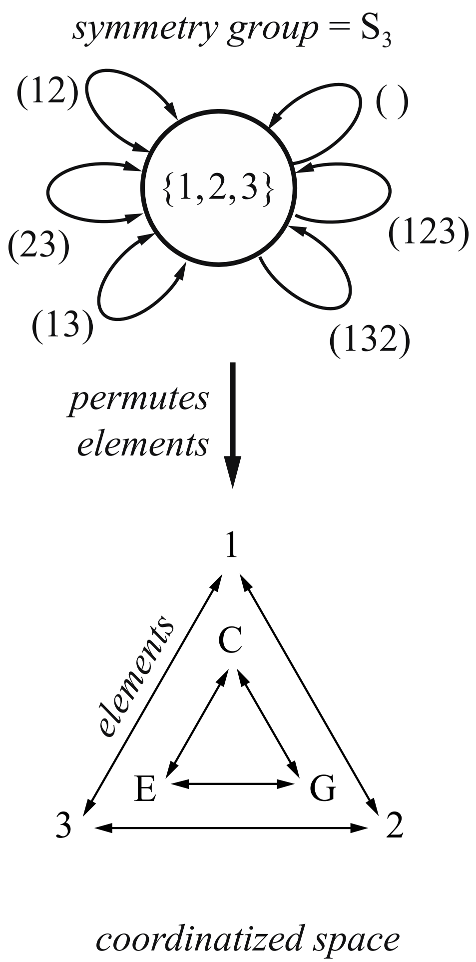
- Attributes generalize the notion of a coordinate system.
  - or perhaps: substance/accident
- Permutations are only conceivable if we have a *coordinatized space*: a mapping between objects and coordinates.
- A very general, mathematical "principle of relativity."
- As we will see, they link the subjective and the objective.

EXAMPLE: PERMUTATIONS

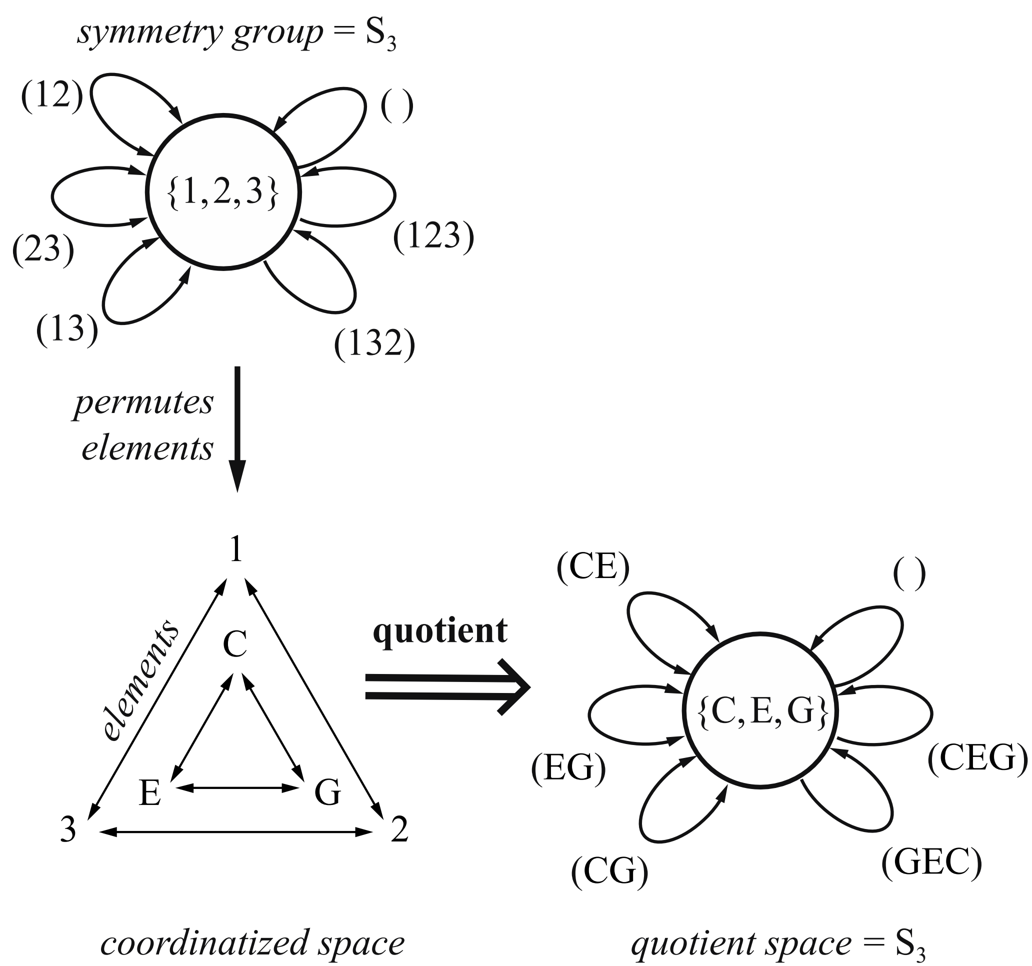


coordinatized space

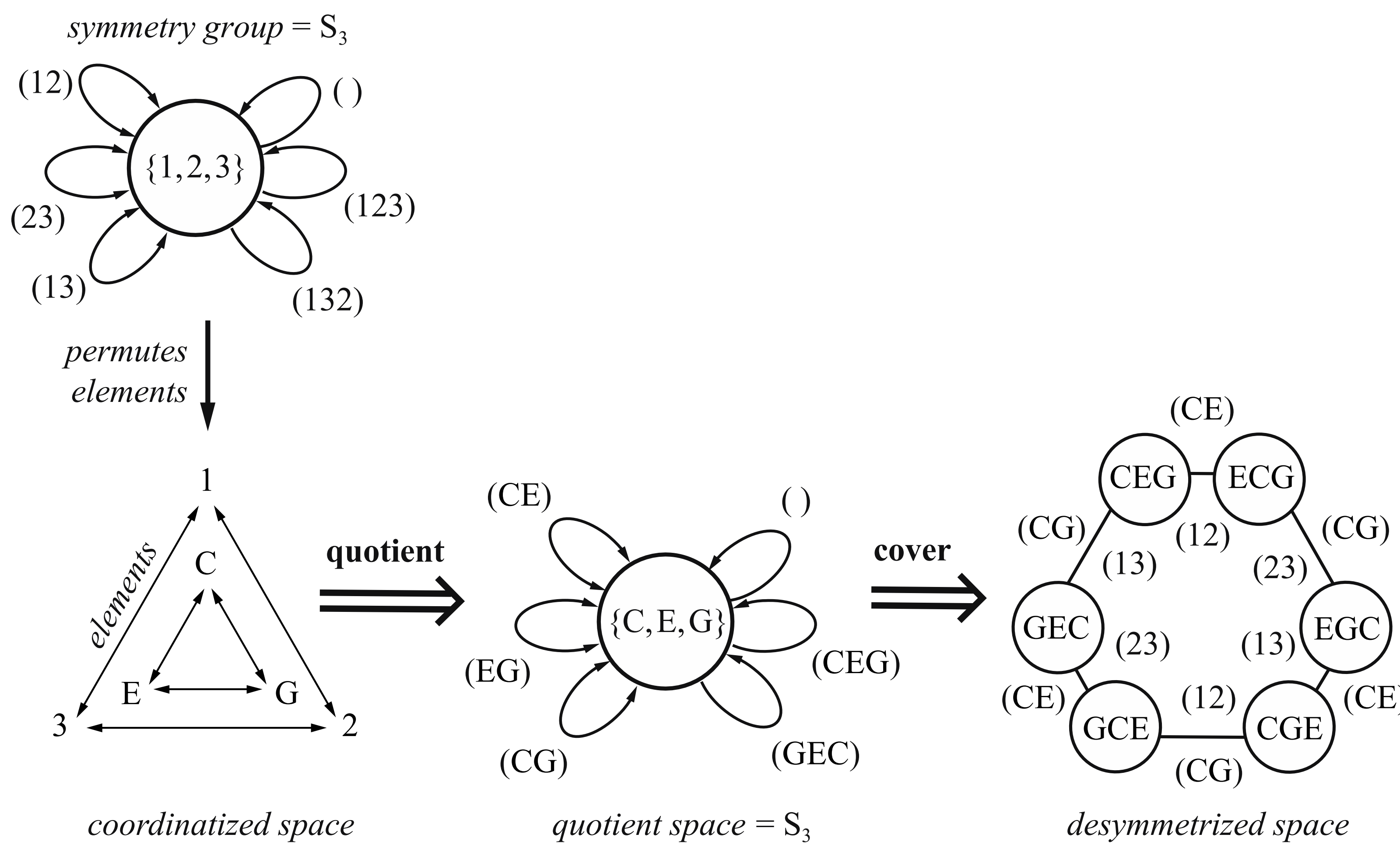
EXAMPLE: PERMUTATIONS



EXAMPLE: PERMUTATIONS

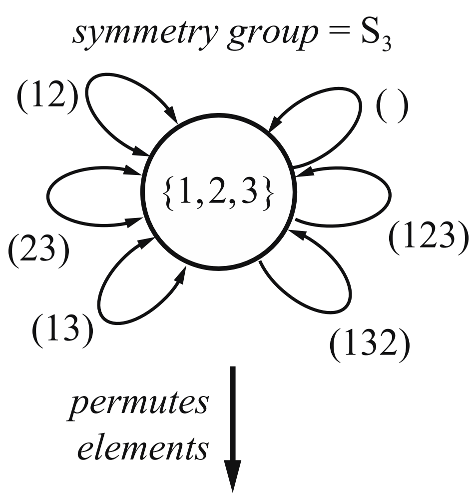


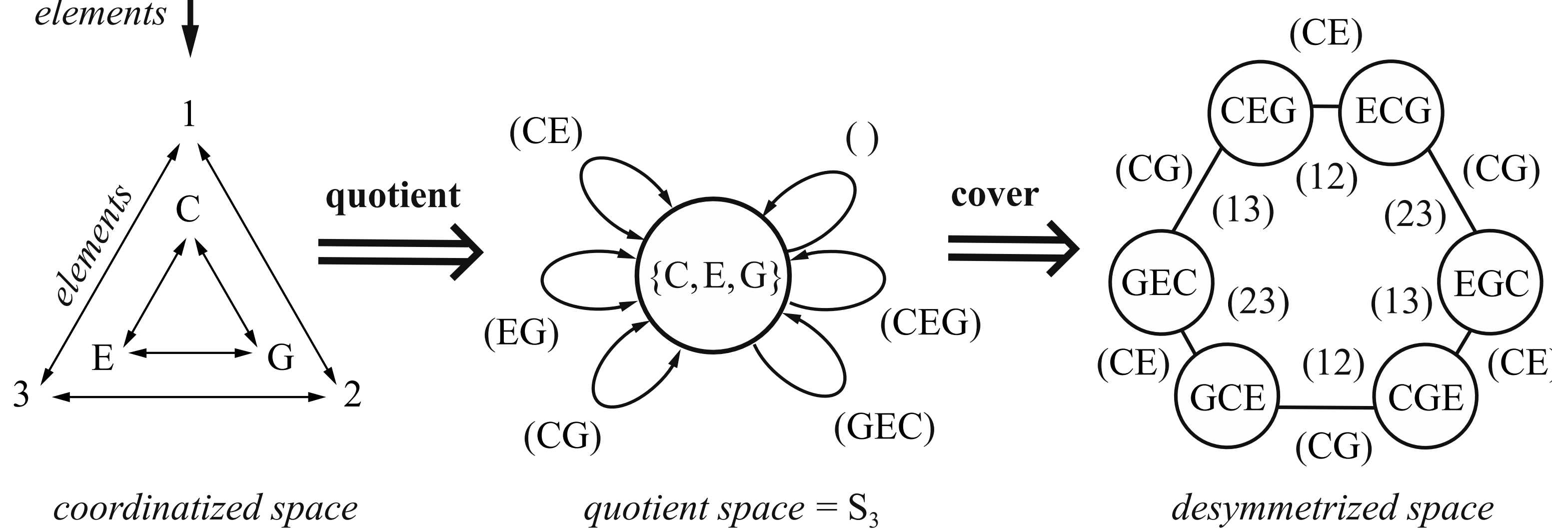
EXAMPLE: PERMUTATIONS



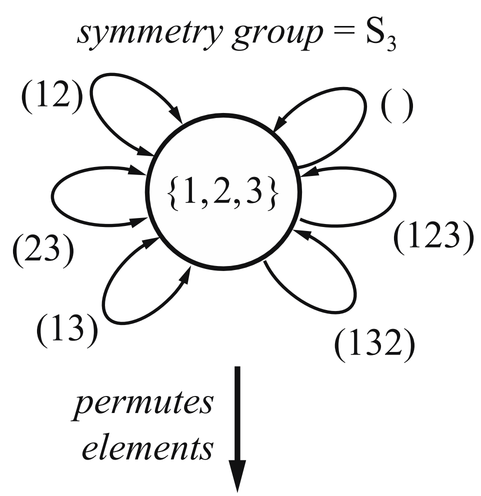
EXAMPLE: PERMUTATIONS

attributes generalize the notion of a coordinate system

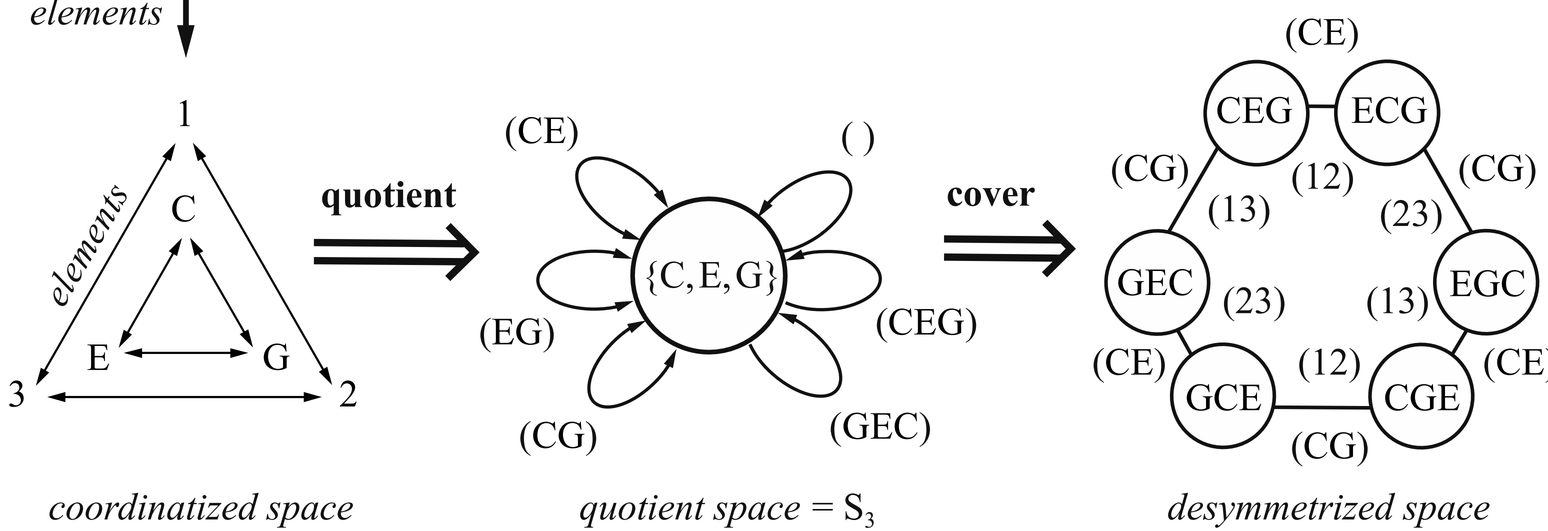




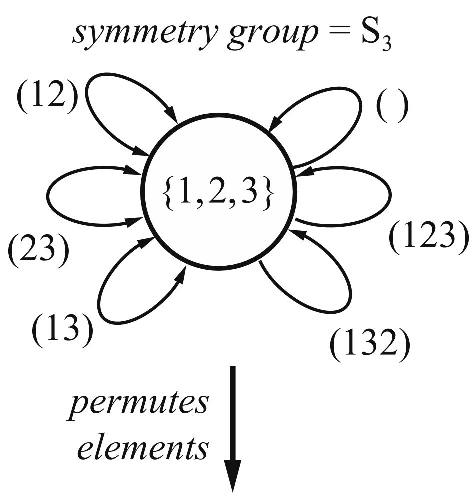
EXAMPLE: PERMUTATIONS



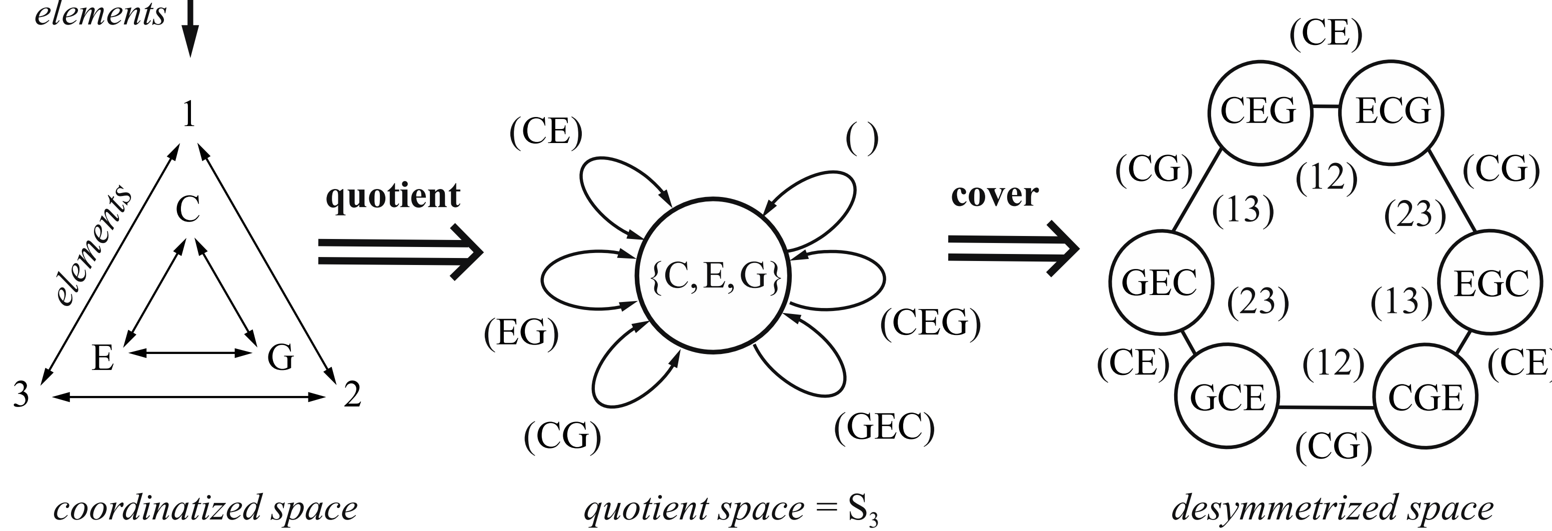
loops in the quotient permute objects the way symmetries permute coordinates



EXAMPLE: PERMUTATIONS



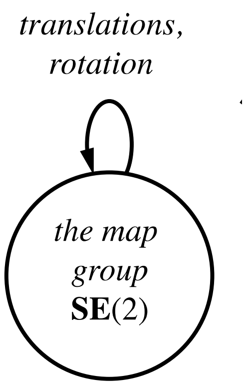
loops in the quotient permute objects the way symmetries permute coordinates ("active" vs. "passive")



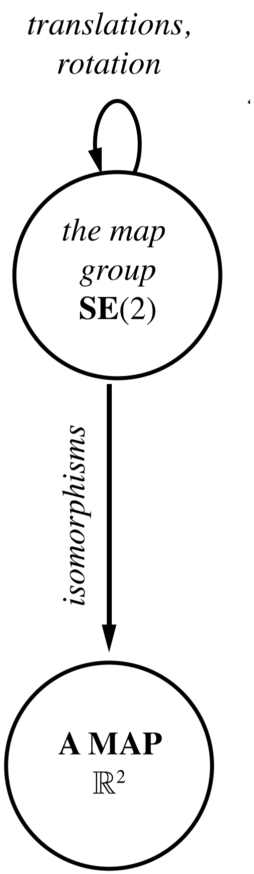
TWO KINDS OF TRANSFORMATION

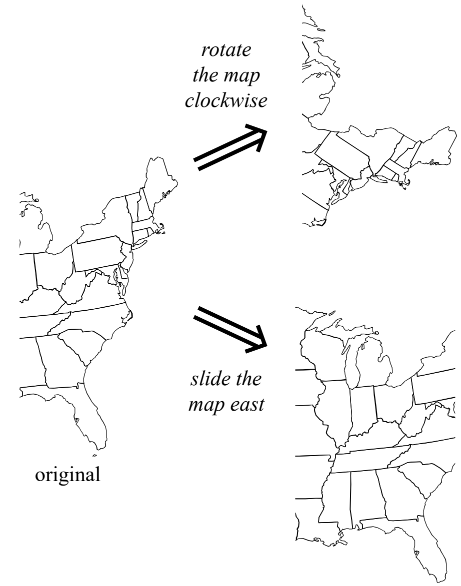
- Element permutations provide an *objective, observer-independent* set of transformations on a space ("Cartesian").
- Attribute permutations can be associated with a *subjective,* observer-relative perspective on the space ("transformational").
- The presence of both attributes and elements allows us to define a generalized theory of symmetry.

EXAMPLE: MAPS

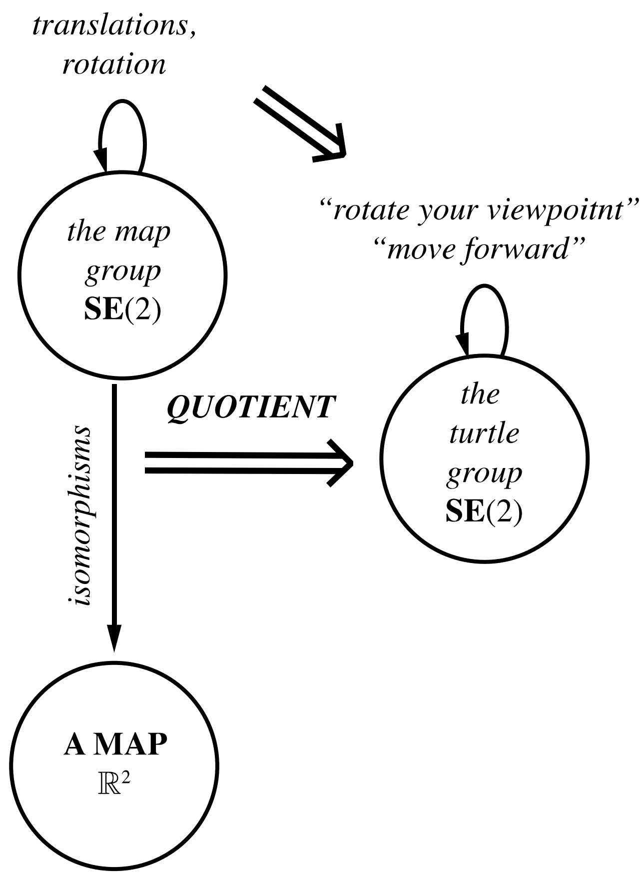


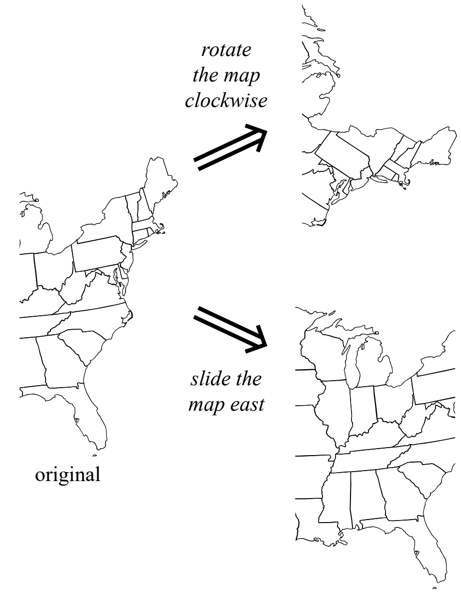
EXAMPLE: MAPS

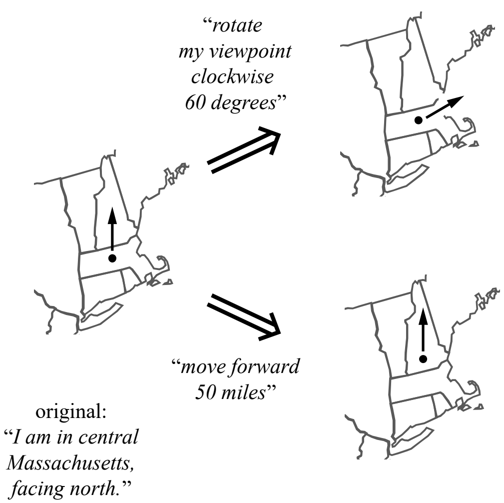




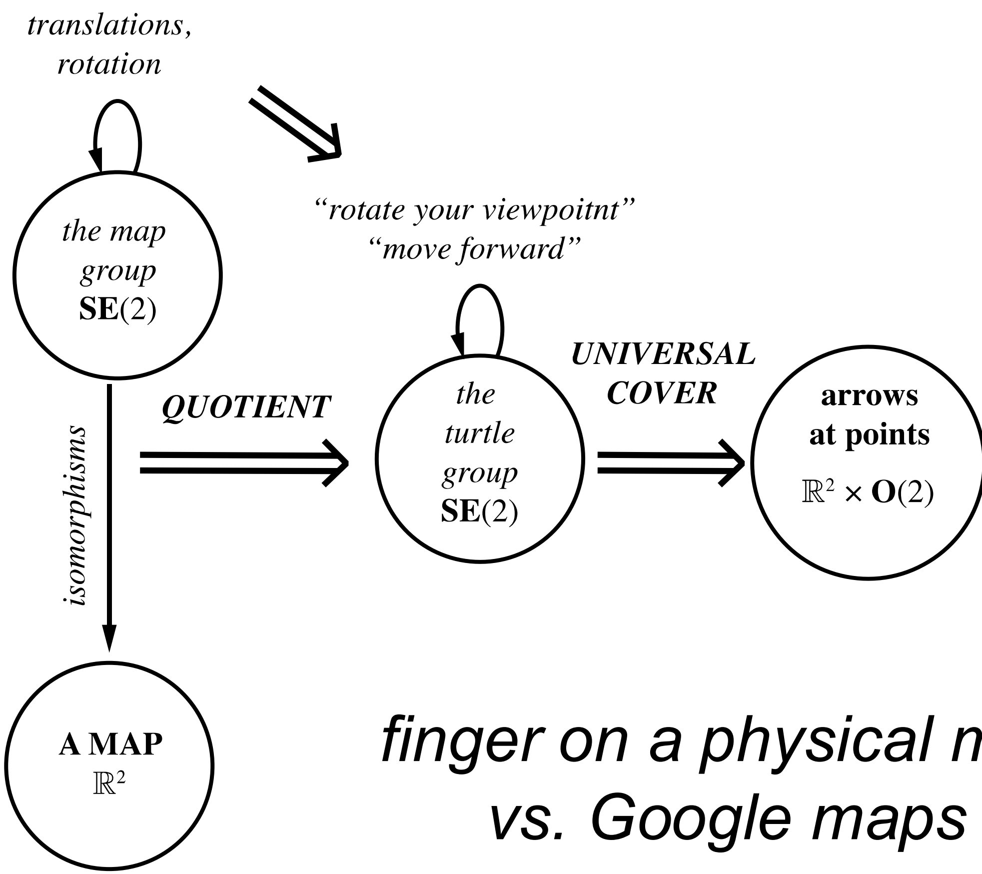
EXAMPLE: MAPS

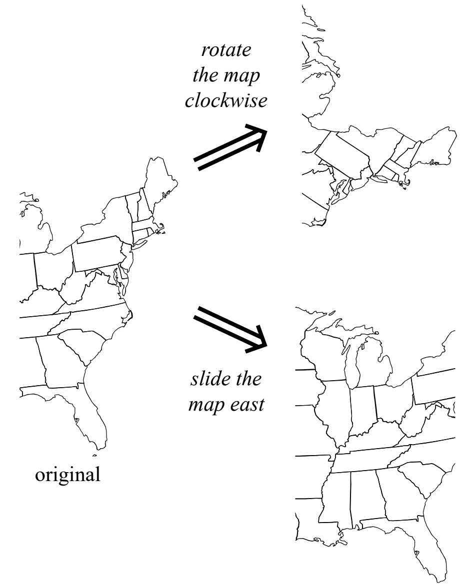


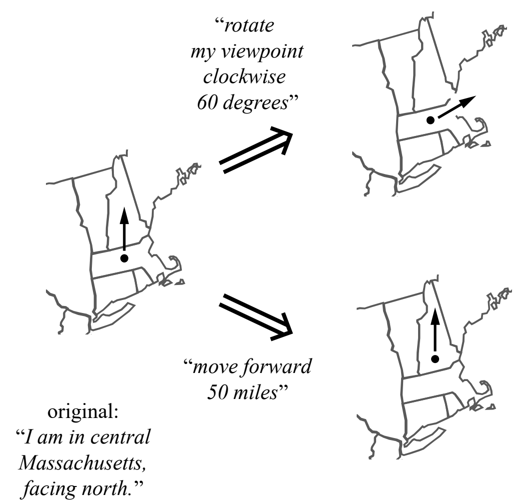




EXAMPLE: MAPS







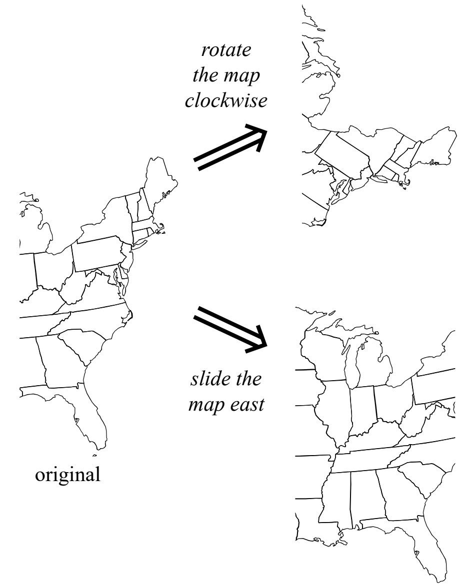
finger on a physical map
vs. Google maps

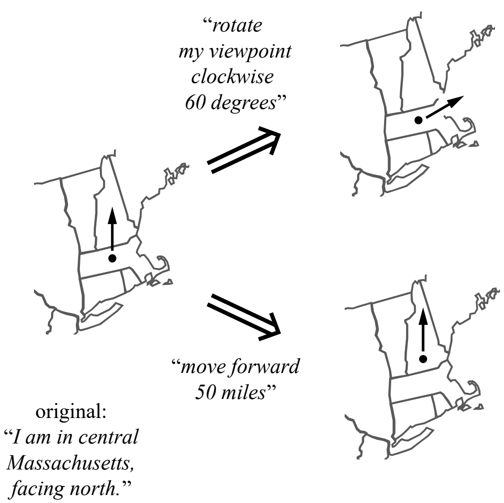
EXAMPLE: MAPS

*Coordinates can represent subject-centered **perspectives** on a space.*

The left and right actions correspond to objective and subjective transformations of the space.

These are isomorphic.



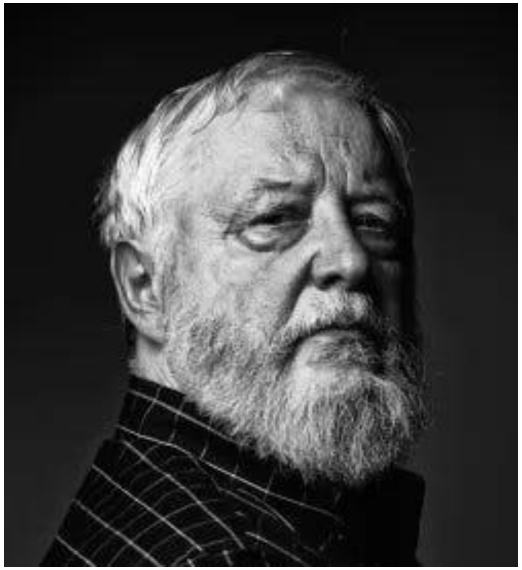


UNIVERSALITY

- This structure is universal
  - can be defined whenever any symmetry group acts on any set or space (= groupoid).
- Any symmetry group gives rise to a quotient space.
  - loops in the quotient space represent a subjective perspective
  - apply symmetries in reverse-chronological order
- Provides a geometrical model of a certain kind of concept-formation (generalization into categories)

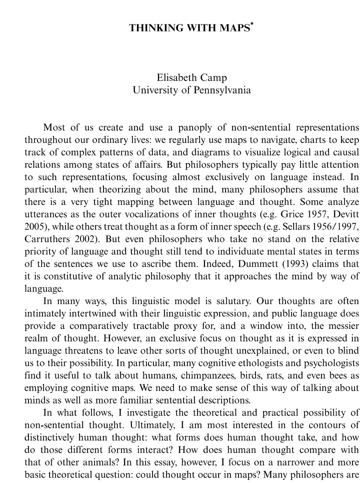
LAWVERE

*The technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as [...]**objective vs. subjective.***



CONECTIONS TO PHILOSOPHY

- Elisabeth Camp (my spouse) is a prominent English-speaking philosopher working on maps and perspectives.
- *Indexicality* is a key component of Camp’s work.
  - requires more than just a space
  - a privileged location and direction (“you are here, looking there”)
  - “spatial perspective” = observer-centered distances and directions.



THINKING WITH MAPS*

Elisabeth Camp
University of Pennsylvania

Most of us create and use a panoply of non-sentential representations throughout our ordinary lives: we regularly use maps to navigate, charts to keep track of complex patterns of data, and diagrams to visualize logical and causal relations among states of affairs. But philosophers typically pay little attention to such representations, focusing almost exclusively on language instead. In particular, when theorizing about the mind, many philosophers assume that there is a very tight mapping between language and thought. Some analyze utterances as the outer vocalizations of inner thoughts (e.g. Grice 1957, Devitt 2005), while others treat thought as a form of inner speech (e.g. Sellars 1956/1997, Carruthers 2002). But even philosophers who take no stand on the relative priority of language and thought still tend to individuate mental states in terms of the sentences we use to ascribe them. Indeed, Dummett (1993) claims that it is constitutive of analytic philosophy that it approaches the mind by way of language.

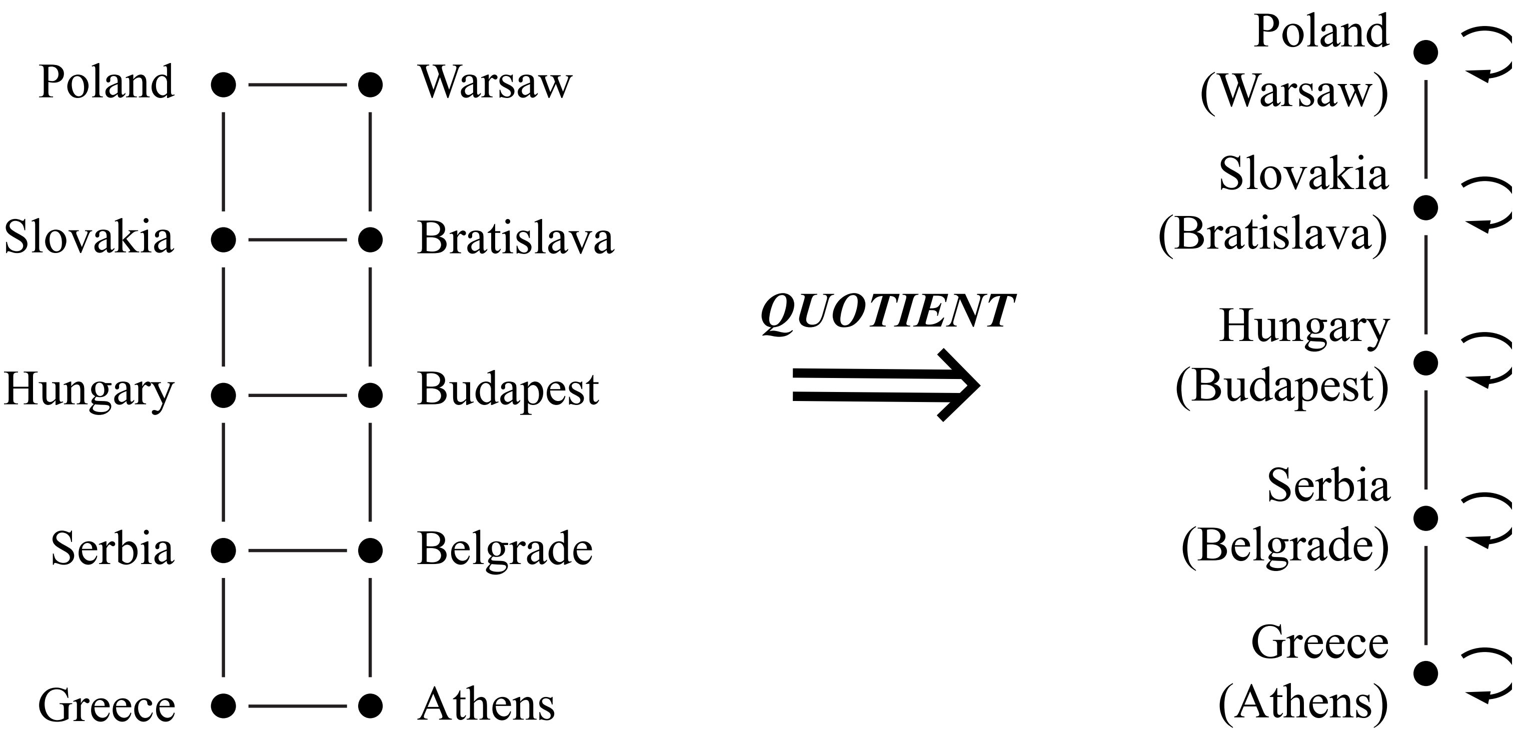
In many ways, this linguistic model is salutary. Our thoughts are often intimately intertwined with their linguistic expression, and public language does provide a comparatively tractable proxy for, and a window into, the messier realm of thought. However, an exclusive focus on thought as it is expressed in language threatens to leave other sorts of thought unexplained, or even to blind us to their possibility. In particular, many cognitive ethologists and psychologists find it useful to talk about humans, chimpanzees, birds, rats, and even bees as employing cognitive maps. We need to make sense of this way of talking about minds as well as more familiar sentential descriptions.

In what follows, I investigate the theoretical and practical possibility of non-sentential thought. Ultimately, I am most interested in the contours of distinctively human thought: what forms does human thought take, and how do those different forms interact? How does human thought compare with that of other animals? In this essay, however, I focus on a narrower and more basic theoretical question: could thought occur in maps? Many philosophers are convinced that in some important sense, thought *per se* must be language-like:

CONNECTIONS TO AI

- In music and philosophy, we deal with quotients of surveyable, low-dimensional spaces.
- It is possible that AI works because it is forming quotients in extremely high-dimensional spaces far beyond the reach of human comprehension.

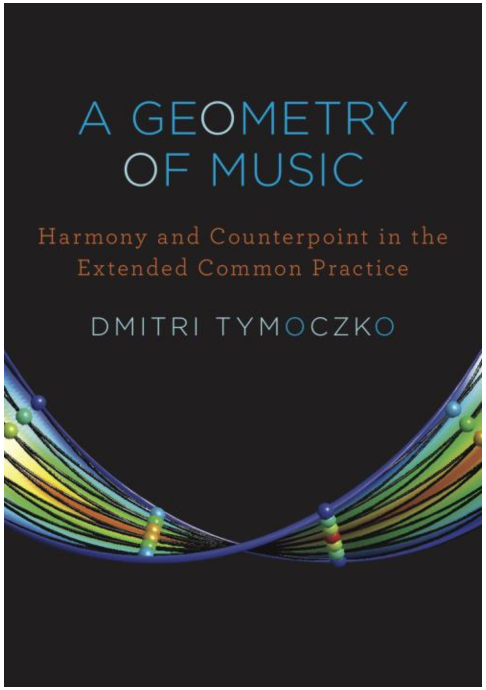
METAPHORICAL PICTURE

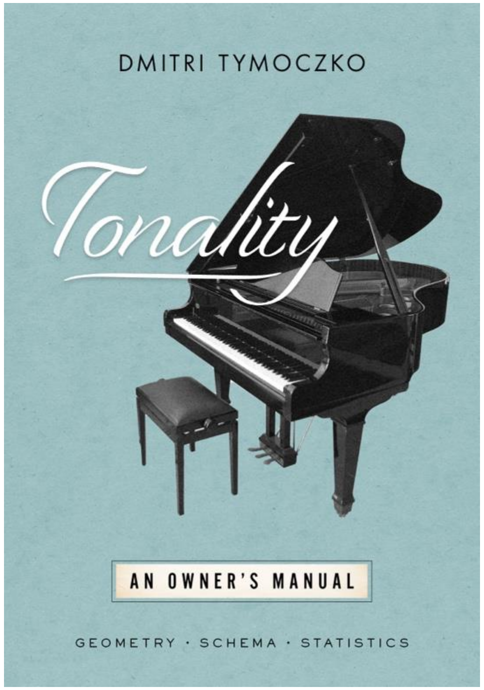


10 objects, 100 arrows

5 objects, 50 arrows

Thank you!





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THANKS TO

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