Imaginary Numbers are not Real * — the Geometric Algebra of Spacetime†

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Abstract

This paper contains a tutorial introduction to the ideas of geometric algebra, concentrating on its physical applications. We show how the definition of a 'geometric product' of vectors in 2- and 3-dimensional space provides precise geometrical interpretations of the imaginary numbers often used in conventional methods. Reflections and rotations are analysed in terms of bilinear spinor transformations, and are then related to the theory of analytic functions and their natural extension in more than two dimensions (monogenics). Physics is greatly facilitated by the use of Hestenes' spacetime algebra, which automatically incorporates the geometric structure of spacetime. This is demonstrated by examples from electromagnetism. In the course of this purely classical exposition many surprising results are obtained — results which are usually thought to belong to the preserve of quantum theory. We conclude that geometric algebra is the most powerful and general language available for the development of mathematical physics.

1 Introduction

...for geometry, you know, is the gate of science, and the gate
is so low and small that one can only enter it as a little child.

William K. Clifford

This paper was commissioned to chronicle the impact that David Hestenes' work has had on physics. Sadly, it seems to us that his work has so far not really had the impact it deserves to have, and that what is needed in this volume is that his message be AMPLIFIED and stated in a language that ordinary physicists understand. With his background in philosophy

*The title of this paper is inspired by David Hestenes, who is known to have a fondness for deliberate ambiguity[1].

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and mathematics, David is certainly no ordinary physicist, and we have observed that his ideas are a source of great mystery and confusion to many[2]. David accurately described the typical response when he wrote[3] that '*physicists quickly become impatient with any discussion of elementary concepts*' — a phenomenon we have encountered ourselves.

We believe that there are two aspects of Hestenes' work which physicists should take particularly seriously. The first is that the *geometric algebra of spacetime* is the best available mathematical tool for theoretical physics, classical or quantum[4, 5, 6]. Related to this part of the programme is the claim that *complex numbers* arising in physical applications usually have a natural geometric interpretation that is hidden in conventional formulations[5, 7, 8, 9]. David's second major idea is that the Dirac theory of the electron contains important geometric information[1, 3, 10, 11], which is disguised in conventional matrix-based approaches. We hope that the importance and truth of this view will be made clear in this and the three following papers. As a further, more speculative, line of development, the hidden geometric content of the Dirac equation has led David to propose a more detailed model of the motion of an electron than is given by the conventional expositions of quantum mechanics. In this model[12, 13], the electron has an electromagnetic field attached to it, oscillating at the 'zitterbewegung' frequency, which acts as a physical version of the de Broglie pilot-wave[14].

David Hestenes' willingness to ask the sort of question that Feynman specifically warned against[1], and to engage in varying degrees of speculation, has undoubtedly had the unfortunate effect of diminishing the impact of his first idea, that geometric algebra can provide a unified language for physics — a contention that we strongly believe. In this paper, therefore, we will concentrate on the first aspect of David's work, deferring to a companion paper[16] any critical examination of his interpretation of the Dirac equation.

In Section 2 we provide a gentle introduction to geometric algebra, emphasising the geometric meaning of the associative (Clifford) product of vectors. We illustrate this with the examples of 2- and 3-dimensional space, showing that it is possible to interpret the unit scalar imaginary number as arising from the geometry of *real* space. Section 3 introduces the powerful techniques by which geometric algebra deals with rotations. This leads to a discussion of the role of spinors in physics. In Section 4 we outline the vector calculus in geometric algebra and review the subject of monogenic functions; these are higher-dimensional generalisations of the analytic functions of two dimensions. Relativity is introduced in Section 5, where we show how Maxwell's equations can be combined into a single relation in geometric algebra, and give a simple general formula for the electromagnetic field of an accelerating charge. We conclude by comparing geometric algebra with alternative languages currently popular in physics. The paper is based on an lecture given by one of us (SFG) to an audience containing both students and professors. Thus, only a modest level of mathematical sophistication (though an open mind) is required to follow it. We nevertheless hope that physicists will find in it a number of surprises; indeed we hope that they will be surprised that there are so many surprises!

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[1]'Do not keep saying to yourself, if you can possibly avoid it, 'But how can it be like that?', because you will get "down the drain", into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that.'[15].

2 An Outline of Geometric Algebra

The new math — so simple only a child can do it. *Tom Lehrer*

Our involvement with David Hestenes began ten years ago, when he attended a Maximum Entropy conference in Laramie. It is a testimony to David's range of interests that one of us (SFG) was able to interact with him at conferences for the next six years, without becoming aware of his interests outside the fields of MaxEnt[17], neural research[18] and the teaching of physics[19]. He apparently knew that astronomers would not be interested in geometric algebra. Our infection with his ideas in this area started in 1988, when another of us (ANL) stumbled across David's book 'Space-Time Algebra'[20], and became deeply impressed. In that summer, our annual MaxEnt conference was in Cambridge, and contact was finally made. Even then, two more months passed before our group reached the critical mass of having two people in the same department, as a result of SFG's reading of David's excellent summary 'A Unified Language for Mathematics and Physics'[5]. Anyone who is involved with Bayesian probability or MaxEnt is accustomed to the polemical style of writing, but his 6-page introduction on the deficiencies of our mathematics is strong stuff. In summary, David said that physicists had not learned properly how to multiply vectors and, as a result of attempts to overcome this, had evolved a variety of mathematical systems and notations that has come to resemble Babel. Four years on, having studied his work in more detail, we believe that he wrote no less than the truth and that, as a result of learning how to multiply vectors together, we can all gain a great increase in our mathematical facility and understanding.

2.1 How to Multiply Vectors

A linear space is one upon which addition and scalar multiplication are defined. Although such a space is often called a 'vector space', our use of the term 'vector' will be reserved for the geometric concept of a *directed line segment*. We still require linearity, so that for any vectors $\mathbf{a}$ and $\mathbf{b}$ we must be able to define their vector sum $\mathbf{a}+\mathbf{b}$. Consistent with our purpose, we will restrict scalars to be real numbers, and define the product of a scalar λ and a vector $\mathbf{a}$ as $\lambda\mathbf{a}$. We would like this to have the geometrical interpretation of being a vector 'parallel' to $\mathbf{a}$ and of 'magnitude' λ times the 'magnitude' of $\mathbf{a}$. To express algebraically the geometric idea of magnitude, we require that an *inner product* be defined for vectors.

- The *inner product* $\mathbf{a}\cdot\mathbf{b}$, also known as the dot or scalar product, of two vectors $\mathbf{a}$ and $\mathbf{b}$, is a *scalar* with magnitude $|\mathbf{a}||\mathbf{b}|\cos\theta$, where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the lengths of $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle between them. Here $|\mathbf{a}| \equiv (\mathbf{a}\cdot\mathbf{a})^{\frac{1}{2}}$, so that the expression for $\mathbf{a}\cdot\mathbf{b}$ is effectively an algebraic definition of $\cos\theta$.

This product contains partial information about the relative direction of any two vectors, since it vanishes if they are perpendicular. In order to capture the remaining information about direction, another product is conventionally introduced, the vector cross product.

- The *cross product* $\mathbf{a}\times\mathbf{b}$ of two vectors is a *vector* of magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$ in the direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ form a right-handed set.

2.2 A Little Un-Learning

These products of vectors, together with their expressions in terms of components (which we will not need or use here), form the basis of everyday teaching in mathematical physics. In fact, the vector cross product is an accident of our 3-dimensional world; in two dimensions there simply isn't a direction perpendicular to $\mathbf{a}$ and $\mathbf{b}$, and in four or more dimensions that direction is ambiguous. A more general concept is needed, so that full information about relative directions can still be encoded in all dimensions. Thus, we will temporarily un-learn the cross product, and instead introduce a new product, called the *outer product*:

- The outer product $\mathbf{a}\wedge\mathbf{b}$ has magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$, but is not a scalar or a vector; it is a *directed area*, or *bivector*, oriented in the plane containing $\mathbf{a}$ and $\mathbf{b}$. The outer product has the same magnitude as the cross product and shares its anticommutative (skew) property: $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$.

A way to visualise the outer product is to imagine $\mathbf{a}\wedge\mathbf{b}$ as the area 'swept out' by displacing $\mathbf{a}$ along $\mathbf{b}$, with the orientation given by traversing the parallelogram so formed first along an $\mathbf{a}$ vector then along a $\mathbf{b}$ vector[4]. This notion leads to a generalisation (due to Grassmann[21]) to products of objects with higher dimensionality, or *grade*. Thus, if the bivector $\mathbf{a}\wedge\mathbf{b}$ (grade 2) is swept out along another vector $\mathbf{c}$ (grade 1), we obtain the directed volume element $(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}$, which is a trivector (grade 3). By construction, the outer product is associative:

$$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = \mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}. \tag{2.1}$$

We can go no further in 3-dimensional space — there is nowhere else to go. Correspondingly, the outer product of any four vectors $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}$ is zero.

At this point we also drop the convention of using bold-face type for vectors such as $\mathbf{a}$ — henceforth vectors and all other grades will be written in ordinary type (with one specific exception, discussed below).

2.3 The Geometric Product

The inner and outer products of vectors are not the whole story. Since $a\cdot b$ is a scalar and $a\wedge b$ is a bivector area, the inner and outer products respectively lower and raise the grade of a vector. They also have opposite commutation properties:

$$\begin{aligned} a\cdot b &= b\cdot a, \\ a\wedge b &= -b\wedge a. \end{aligned} \tag{2.2}$$

In this sense we can think of the inner and outer products together as forming the symmetric and antisymmetric parts of a new product, (defined originally by Grassmann[22] and Clifford[23]) which we call the *geometric product*, ab:

$$ab = a\cdot b + a\wedge b. \tag{2.3}$$

Thus, the product of parallel vectors is a scalar — we take such a product, for example, when finding the length of a vector. On the other hand, the product of orthogonal vectors is a bivector — we are finding the directed area of something. It is reasonable to suppose that the product of vectors that are neither parallel nor perpendicular should contain both scalar and bivector parts.

How on Earth do I Add a Scalar to a Bivector?

Most physicists need a little help at this point[2]. Adding together a scalar and a bivector doesn't seem right at first — they are different types of quantities. But it is exactly what you want an addition to do! The result of adding a scalar to a bivector is an object that has both scalar and bivector parts, in exactly the same way that the addition of real and imaginary numbers yields an object with both real and imaginary parts. We call this latter object a 'complex number' and, in the same way, we shall refer to a (scalar + bivector) as a 'multivector', accepting throughout that we are combining objects of different types. The addition of scalar and bivector does not result in a single new quantity in the same way as $2+3=5$; we are simply keeping track of separate components in the symbol $ab = a\cdot b + a\wedge b$ or $z = x + iy$. This type of addition, of objects from separate linear spaces, could be given the symbol $\oplus$, but it should be evident from our experience of complex numbers that it is harmless, and more convenient, to extend the definition of addition and use the plain, ordinary $+$ sign.

We have defined the geometric product in terms of the inner and outer product of two vectors. An alternative and more mathematical approach is to define the associative geometric product via a set of axioms and introduce two 'new' products $a\cdot b \equiv \frac{1}{2}(ab+ba)$ and $a\wedge b \equiv \frac{1}{2}(ab-ba)$. Then, for example, if we assert that the square of any vector should be a scalar, this would allow us to prove that the product $a\cdot b$ is scalar-valued, since $ab+ba = (a+b)^2 - a^2 - b^2$. This more axiomatic approach is taken in Chapter 1 of Hestenes & Sobczyk[6].

2.4 Geometric Algebra of the Plane

A 1-dimensional space has insufficient geometric structure to show what is going on, so we begin in two dimensions, taking two orthonormal basis vectors σ_1 and σ_2. These satisfy the relations

$$\begin{aligned} \sigma_1\cdot\sigma_1 &= 1, \qquad & \sigma_1\wedge\sigma_1 &= 0, \\ \sigma_2\cdot\sigma_2 &= 1, \qquad & \sigma_2\wedge\sigma_2 &= 0 \end{aligned} \tag{2.4}$$

and

$$\sigma_1\cdot\sigma_2 = 0. \tag{2.5}$$

The outer product $\sigma_1\wedge\sigma_2$ represents the directed area element of the plane and we assume that σ_1, σ_2 are chosen such that this has the conventional right-handed orientation. This completes the geometrically meaningful quantities that we can make from these basis vectors:

$$\begin{array}{ccc} 1, & \{\sigma_1, \sigma_2\}, & \sigma_1\wedge\sigma_2. \\ \text{scalar} & \text{vectors} & \text{bivector} \end{array} \tag{2.6}$$

We now assemble a *Clifford algebra* from these quantities. An arbitrary linear sum over the four basis elements in (2.6) is called a *multivector*. In turn, given two multivectors A and B, we can form their sum $S = A + B$ by adding the components:

$$\begin{array}{ccccccccc} A & \equiv & a_0 1 & + & a_1\sigma_1 & + & a_2\sigma_2 & + & a_3\sigma_1\wedge\sigma_2 \\ B & \equiv & b_0 1 & + & b_1\sigma_1 & + & b_2\sigma_2 & + & b_3\sigma_1\wedge\sigma_2 \\ S & = & (a_0+b_0)1 & + & (a_1+b_1)\sigma_1 & + & (a_2+b_2)\sigma_2 & + & (a_3+b_3)\sigma_1\wedge\sigma_2. \end{array} \tag{2.7}$$

By this definition of a linear sum we have done almost nothing — the power comes from the definition of the multiplication $P = AB$. In order to define this product, we have to be able to multiply the 4 geometric basis elements. Multiplication by a scalar is obvious. To form the products of the vectors we remember the definition $ab = a\cdot b + a\wedge b$, so that

$$\begin{aligned} \sigma_1^2 = \sigma_1\sigma_1 = \sigma_1\cdot\sigma_1 + \sigma_1\wedge\sigma_1 = 1 = \sigma_2^2, \\ \sigma_1\sigma_2 = \sigma_1\cdot\sigma_2 + \sigma_1\wedge\sigma_2 = \sigma_1\wedge\sigma_2 = -\sigma_2\sigma_1. \end{aligned} \tag{2.8}$$

Products involving the bivector $\sigma_1\wedge\sigma_2 = \sigma_1\sigma_2$ are particularly important. Since the geometric product is associative, we have:

$$\begin{aligned} (\sigma_1\sigma_2)\sigma_1 &= -\sigma_2\sigma_1\sigma_1 = -\sigma_2, \\ (\sigma_1\sigma_2)\sigma_2 &= \sigma_1 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \sigma_1(\sigma_1\sigma_2) &= \sigma_2 \\ \sigma_2(\sigma_1\sigma_2) &= -\sigma_1. \end{aligned} \tag{2.10}$$

The only other product is the square of $\sigma_1\wedge\sigma_2$:

$$(\sigma_1\wedge\sigma_2)^2 = \sigma_1\sigma_2\sigma_1\sigma_2 = -\sigma_1\sigma_1\sigma_2\sigma_2 = -1. \tag{2.11}$$

These results complete the definition of the product and enable, for example, the processes of addition and multiplication to be coded as computer functions. In principle, these definitions could be made an intrinsic part of a computer language, in the same way that complex number arithmetic is already intrinsic to some languages. To reinforce this point, it may be helpful to write out the product explicitly. We have,

$$AB = P \equiv p_0 1 + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_1\wedge\sigma_2, \tag{2.12}$$

where

$$\begin{aligned} p_0 &= a_0b_0 + a_1b_1 + a_2b_2 - a_3b_3, \\ p_1 &= a_0b_1 + a_1b_0 + a_3b_2 - a_2b_3, \\ p_2 &= a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1, \\ p_3 &= a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1. \end{aligned} \tag{2.13}$$

Multivector addition and multiplication obey the associative and distributive laws, so that we have, as promised, the geometric *algebra* of the plane.

We emphasise the important features that have emerged in the course of this derivation.

- The geometric product of two parallel vectors is a scalar number — the product of their lengths.
- The geometric product of two perpendicular vectors is a bivector — the directed area formed by the vectors.
- Parallel vectors commute under the geometric product; perpendicular vectors anticommute.
- The bivector $\sigma_1\wedge\sigma_2$ has the geometric effect of *rotating* the vectors $\{\sigma_1, \sigma_2\}$ in its own plane by 90° clockwise when multiplying them on their left. It rotates vectors by 90° anticlockwise when multiplying on their right. This can be used to define the orientation of σ_1 and σ_2.

- The square of the bivector area $\sigma_1 \wedge \sigma_2$ is a scalar: $(\sigma_1 \wedge \sigma_2)^2 = -1$.

By virtue of the last two properties the bivector $\sigma_1 \wedge \sigma_2$ becomes our first candidate for the role of the unit imaginary i, and in 2-dimensional applications it fulfills this role admirably. Indeed, we see that the even-grade elements $z = x + y\sigma_1\sigma_2$ form a natural subalgebra, equivalent to the *complex numbers*.

2.5 The Algebra of 3-Space

If we now add a third orthonormal vector σ_3 to our basis set, we generate the following geometrical objects:

$$\begin{array}{cccc} 1, & \{\sigma_1, \sigma_2, \sigma_3\}, & \{\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1\}, & \sigma_1\sigma_2\sigma_3 \\ \text{scalar} & \text{3 vectors} & \text{3 bivectors} & \text{trivector} \\ & & \text{area elements} & \text{volume element} \end{array} \tag{2.14}$$

From these objects we form a linear space of $(1 + 3 + 3 + 1) = 8 = 2^3$ dimensions, defining multivectors as before, together with the operations of addition and multiplication. Most of the algebra is the same as in the 2-dimensional version because the subsets $\{\sigma_1, \sigma_2\}$, $\{\sigma_2, \sigma_3\}$ and $\{\sigma_3, \sigma_1\}$ generate 2-dimensional subalgebras, so that the only new geometric products we have to consider are

$$\begin{aligned} (\sigma_1\sigma_2)\sigma_3 &= \sigma_1\sigma_2\sigma_3, \\ (\sigma_1\sigma_2\sigma_3)\sigma_k &= \sigma_k(\sigma_1\sigma_2\sigma_3) \end{aligned} \tag{2.15}$$

and

$$(\sigma_1\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_3^2 = -1. \tag{2.16}$$

These relations lead to new geometrical insights:

- A simple bivector rotates vectors in its own plane by 90°, but forms trivectors (volumes) with vectors perpendicular to it.

- The trivector $\sigma_1\sigma_2\sigma_3$ commutes with all vectors, and hence with all multivectors.

The trivector $\sigma_1\sigma_2\sigma_3$ also has the algebraic property of being a square root of minus one. In fact, of the eight geometrical objects, four have negative square $\{\sigma_1\sigma_2,\ \sigma_2\sigma_3,\ \sigma_3\sigma_1,\ \sigma_1\sigma_2\sigma_3\}$. Of these, the trivector $\sigma_1\sigma_2\sigma_3$ is distinguished by its commutation properties, and by the fact that it is the highest-grade element in the space. Highest-grade objects are generically called *pseudoscalars*, and $\sigma_1\sigma_2\sigma_3$ is thus the unit pseudoscalar for 3-dimensional space. In view of its properties we give it the special symbol i:

$$i \equiv \sigma_1\sigma_2\sigma_3. \tag{2.17}$$

We should be quite clear, however, that we are using the symbol i to stand for a pseudoscalar, and thus cannot use the same symbol for the commutative scalar imaginary, as used for example in conventional quantum mechanics, or in electrical engineering. We shall use the symbol j for this uninterpreted imaginary, consistent with existing usage in engineering. The definition (2.17) will be consistent with our later extension to 4-dimensional spacetime.

2.6 Interlude

We have now reached the point which is liable to cause the greatest intellectual shock. We have played an apparently harmless game with the algebra of 3-dimensional vectors and found a geometric quantity $i \equiv \sigma_1\sigma_2\sigma_3$ which has negative square and commutes with all multivectors. Multiplying this by σ_3, σ_1 and σ_2 in turn we get

$$\begin{aligned} (\sigma_1\sigma_2\sigma_3)\sigma_3 = \sigma_1\sigma_2 &= i\sigma_3, \\ \sigma_2\sigma_3 &= i\sigma_1, \\ \sigma_3\sigma_1 &= i\sigma_2, \end{aligned} \tag{2.18}$$

which is exactly the algebra of the Pauli spin matrices used in the quantum mechanics of spin-$\frac{1}{2}$ particles! The familiar Pauli matrix relation,

$$\hat{\sigma}_i\hat{\sigma}_j = 1\delta_{ij} + j\epsilon_{ijk}\hat{\sigma}_k, \tag{2.19}$$

is now nothing more than an expression of the geometric product of orthonormal vectors. We shall demonstrate the equivalence with the Pauli matrix algebra explicitly in a companion paper[24], but here it suffices to note that the matrices

$$\hat{\sigma}_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 \equiv \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.20}$$

comprise a matrix representation of our 3-dimensional geometric algebra. Indeed, since we can represent our algebra by these matrices, it should now be obvious that we can indeed add together the various different geometric objects in the algebra — we just add the corresponding matrices. These matrices have four complex components (eight degrees of freedom), so we could always disentangle them again.

Now it is clearly true that *any* associative algebra can be represented by a matrix algebra; but that matrix representation may not be the *best* interpretation of what is going on. In the quantum mechanics of spin-$\frac{1}{2}$ particles we have a case where generations of physicists have been taught nothing but matrices, when there is a perfectly good geometrical interpretation of those same equations! And it gets worse. We were taught that the $(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ were the components of a vector $\hat{\boldsymbol{\sigma}}$, and how to write things like $\mathbf{a}\cdot\hat{\boldsymbol{\sigma}} = a_k\hat{\sigma}_k$ and $S^2 = (\hat{\sigma}_1^2+\hat{\sigma}_2^2+\hat{\sigma}_3^2)\hbar^2/4$. But, geometrically, $\{\sigma_1, \sigma_2, \sigma_3\}$ are three orthonormal vectors comprising the basis of space, so that in $a_k\hat{\sigma}_k$ the $\{a_k\}$ are the *components* of a vector along directions σ_k and the result $a_k\sigma_k$ is a *vector*, not a scalar. With regard to S^2, if you want to find the length of a vector, you must square and add the components of the vector along the unit basis vectors — not the basis vectors themselves. So the result $\sigma_k\sigma_k = 3$ is certainly true, but does not have the interpretation usually given to it.

These considerations all indicate that our present thinking about quantum mechanics is infested with the deepest misconceptions. We believe, with David Hestenes, that geometric algebra is an essential ingredient in unravelling these misconceptions.

On the constructive side, the geometric algebra is easy to use, and allows us to manipulate geometric quantities in a coordinate-free way. The σ-vectors, which play an essential role, are thereby removed from the mysteries of quantum mechanics, and used to advantage in physics and engineering. We shall see that a similar fate awaits Dirac's γ-matrices.

The algebra of 3-dimensional space, the Pauli algebra, is central to physics, and deserves further emphasis. It is an 8-dimensional linear space of multivectors, which we write as

$$M = \underset{\text{scalar}}{\alpha} + \underset{\text{vector}}{\mathbf{a}} + \underset{\text{bivector}}{i\mathbf{b}} + \underset{\text{pseudoscalar}}{i\beta} \tag{2.21}$$

where $\mathbf{a} \equiv a_k\sigma_k$, $\mathbf{b} \equiv b_k\sigma_k$, and we have reverted to bold-face type for 3-dimensional vectors. This is the exception referred to earlier; we use this convention[5] to maintain a visible difference between spacetime 4-vectors and vectors of 3-dimensional space. There is never any ambiguity concerning the basis vectors $\{\sigma_k\}$, however, and these will continue to be written unbold.

The space of even-grade elements of this algebra,

$$\psi = \alpha + i\mathbf{b}, \tag{2.22}$$

is closed under multiplication and forms a representation of the quarternion algebra. Explicitly, identifying $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ with $i\sigma_1$, $-i\sigma_2$, $i\sigma_3$, respectively, we have the usual quarternion relations, including the famous formula

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{2.23}$$

Finally in this section, we relearn the cross product in terms of the outer product and duality operation (multiplication by the pseudoscalar):

$$\mathbf{a}\times\mathbf{b} = -i\mathbf{a}\wedge\mathbf{b}. \tag{2.24}$$

Here we have introduced an operator precedence convention in which an outer or inner product always takes precedence over a geometric product. Thus $\mathbf{a}\wedge\mathbf{b}$ is taken before the multiplication by i.

The duality operation in three dimensions interchanges a plane with a vector orthogonal to it (in a right-handed sense). In the mathematical literature this operation goes under the name of the 'Hodge dual'. Quantities like $\mathbf{a}$ or $\mathbf{b}$ would conventionally be called 'polar vectors', while the 'axial vectors' which result from cross-products can now be seen to be disguised versions of *bivectors*.

3 Rotations and Geometric algebra

Geometric algebra is useful to a physicist because it automatically incorporates the structure of the world we inhabit, and accordingly provides a natural language for physics. One of the clearest illustrations of its power is the way in which it deals with reflections and rotations. The key to this approach is a theorem due to Hamilton[25]: given any unit vector n ($n^2 = 1$), we can resolve an arbitrary vector x into parts parallel and perpendicular to n: $x = x_\perp + x_\parallel$. These components are identified algebraically through their commutation properties:

$$\begin{aligned} nx_\parallel &= x_\parallel n \text{ (scalar)}, \\ nx_\perp &= -x_\perp n \text{ (bivector)}. \end{aligned} \tag{3.1}$$

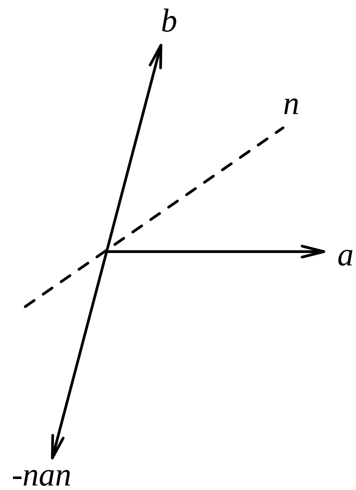

Figure 1: *A rotation composed of two reflections*

The vector $x_\perp - x_{||}$ can therefore be written $-nxn$. Geometrically, the transformation $x \rightarrow -nxn$ represents a *reflection* in a plane perpendicular to n. To make a rotation we need two of these reflections:

$$x \rightarrow mnxnm = Rx\tilde{R}, \tag{3.2}$$

where $R \equiv mn$ is called a 'rotor'. We call $\tilde{R} \equiv nm$ the 'reverse' of R, because it is obtained by reversing the order of all geometric products. The rotor is even (i.e. viewed as a multivector it contains only even-grade elements), and is unimodular, satisfying $R\tilde{R} = \tilde{R}R = 1$.

As an example, let us rotate the unit vector a into another unit vector b, leaving all vectors perpendicular to a and b unchanged (a *simple* rotation). We can accomplish this by a reflection perpendicular to the unit vector which is half-way between a and b (see Figure 1):

$$n \equiv (a+b)/|a+b|. \tag{3.3}$$

This reflects a into $-b$, which we correct by a second reflection perpendicular to b. Algebraically,

$$x \rightarrow b\frac{a+b}{|a+b|}\, x\, \frac{a+b}{|a+b|}b \tag{3.4}$$

which represents the simple rotation in the $a\wedge b$ plane. Since $a^2 = b^2 = 1$, we define

$$R \equiv \frac{1+ba}{|a+b|} = \frac{1+ba}{\sqrt{2(1+b\cdot a)}}, \tag{3.5}$$

so that the rotation is written

$$b = Ra\tilde{R}, \tag{3.6}$$

which is a 'bilinear' transformation of a. The inverse transformation is

$$a = \tilde{R}bR. \tag{3.7}$$

The bilinear transformation of vectors $x \rightarrow Rx\tilde{R}$ is a very general way of handling rotations. In deriving this transformation the dimensionality of the space of vectors was at no point specified. As a result, the transformation law works for *all* spaces, *whatever dimension*. Furthermore, it works for *all* types of geometric object, *whatever grade*. We can see this by considering the product of vectors

$$xy \to Rx\tilde{R}\ Ry\tilde{R} = R(xy)\tilde{R}, \tag{3.8}$$

which holds because $\tilde{R}R = 1$.

As an example, consider a 2-dimensional rotation:

$$e_i = R\sigma_i\tilde{R} \quad (i = 1, 2). \tag{3.9}$$

A rotation by angle θ is performed by the even element (the equivalent of a complex number)

$$R \equiv \exp(-\sigma_1\sigma_2\theta/2) = \cos(\theta/2) - \sigma_1\sigma_2\sin(\theta/2). \tag{3.10}$$

As a check:

$$\begin{aligned} R\sigma_1\tilde{R} &= \exp(-\sigma_1\sigma_2\theta/2)\sigma_1\exp(\sigma_1\sigma_2\theta/2) \\ &= \exp(-\sigma_1\sigma_2\theta)\ \sigma_1 \\ &= \cos\theta\ \sigma_1 + \sin\theta\ \sigma_2, \end{aligned} \tag{3.11}$$

and

$$R\sigma_2\tilde{R} = -\sin\theta\ \sigma_1 + \cos\theta\ \sigma_2. \tag{3.12}$$

The bilinear transformation is much easier to use than the one-sided rotation matrix, because the latter becomes more complicated as the number of dimensions increases. Although this is less evident in two dimensions, in three dimensions it is obvious: the rotor

$$R \equiv \exp(-i\mathbf{a}/2) = \cos(|\mathbf{a}|/2) - i\frac{\mathbf{a}}{|\mathbf{a}|}\sin(|\mathbf{a}|/2) \tag{3.13}$$

represents a rotation of $|\mathbf{a}|$ radians about the axis along the direction of $\mathbf{a}$. If required, we can decompose rotations into Euler angles (θ, ϕ, χ), the explicit form being

$$R = e^{-i\sigma_3\phi/2}e^{-i\sigma_2\theta/2}e^{-i\sigma_3\chi/2}. \tag{3.14}$$

We now examine the composition of rotors in more detail. In three dimensions, let the rotor R transform the unit vector along the z-axis into a vector s:

$$s = R\sigma_3\tilde{R}. \tag{3.15}$$

Now rotate the s vector into another vector s', using a rotor R'. This requires

$$s' = R's\tilde{R}' = (R'R)\sigma_3(R'R)\tilde{}, \tag{3.16}$$

so that the transformation is characterised by

$$R \to R'R, \tag{3.17}$$

which is the (left-sided) group combination rule for rotors. Now suppose that we start with s and make a rotation of 360° about the z-axis, so that s' returns to s. What happens to R is surprising; using (3.13) above, we see that

$$R \to -R. \tag{3.18}$$

This is the behaviour of spin-$\frac{1}{2}$ particles in quantum mechanics, yet we have done nothing quantum-mechanical; we have merely built up rotations from reflections.

How can this be? It turns out[24] that it is possible to represent a Pauli spinor $|\psi\rangle$ (a 2-component complex spinor) as an arbitrary even element ψ (four real components) in the geometric algebra of 3-space (2.21). Since $\psi\tilde{\psi}$ is a positive-definite scalar in the Pauli algebra we can write

$$\psi = \rho^{\frac{1}{2}} R. \tag{3.19}$$

Thus, the Pauli spinor $|\psi\rangle$ can be seen as a (heavily disguised) instruction to rotate and dilate. The identification of a rotor component in ψ then explains the double-sided action of spinors on observables. The spin-vector observable, for example, can be written in geometric algebra as

$$S = \psi\sigma_3\tilde{\psi} = \rho R\sigma_3\tilde{R}, \tag{3.20}$$

which has the same form as equation (3.15). This identification of quantum spin with rotations is very satisfying, and provides much of the impetus for David Hestenes' work on Dirac theory.

A problem remaining is what to call an arbitrary even element ψ. We shall call it a *spinor*, because the space of even elements forms a closed algebra under the left-sided action of the rotation group: $\psi \to R\psi$ gives another even element. This accords with the usual abstract definition of spinors from group representation theory, but differs from the column vector definition favoured by some authors[26].

4 Analytic and Monogenic Functions

Returning to 2-dimensional space, we now use geometric algebra to reveal the structure of the Argand diagram. From any vector $r = x\sigma_1 + y\sigma_2$ we can form an even multivector (a 2-dimensional spinor):

$$z \equiv \sigma_1 r = x + Iy, \tag{4.1}$$

where

$$I \equiv \sigma_1\sigma_2. \tag{4.2}$$

Using the vector σ_1 to define the real axis, there is therefore a one-to-one correspondence between points in the Argand diagram and vectors in two dimensions. Complex conjugation,

$$z^* \equiv \tilde{z} = r\sigma_1 = x - Iy, \tag{4.3}$$

now appears as a natural operation of reversion for the even multivector z, and (as shown above) it is needed when rotating vectors.

We now consider the fundamental derivative operator

$$\nabla \equiv \sigma_1\partial_x + \sigma_2\partial_y, \tag{4.4}$$

and observe that

$$\nabla z = (\sigma_1 \partial_x + \sigma_2 \partial_y)(x + \sigma_1 \sigma_2 y) = \sigma_1 + \sigma_2 \sigma_1 \sigma_2 = 0, \tag{4.5}$$

$$\nabla z^* = (\sigma_1 \partial_x + \sigma_2 \partial_y)(x - \sigma_1 \sigma_2 y) = \sigma_1 - \sigma_2 \sigma_1 \sigma_2 = 2\sigma_1. \tag{4.6}$$

Generalising this behaviour, we find that

$$\nabla z^n = 0, \tag{4.7}$$

and define an *analytic function* as a function $f(z)$ (or, equivalently, $f(r)$) for which

$$\nabla f = 0. \tag{4.8}$$

Writing $f = u + Iv$, this implies that

$$(\partial_x u - \partial_y v)\sigma_1 + (\partial_y u + \partial_x v)\sigma_2 = 0, \tag{4.9}$$

which are the Cauchy-Riemann conditions. It follows immediately that any non-negative, integer power series of z is analytic. The vector derivative is *invertible* so that, if

$$\nabla f = s \tag{4.10}$$

for some function s, we can find f as

$$f = \nabla^{-1} s. \tag{4.11}$$

Cauchy's integral formula for analytic functions is an example of this:

$$f(z) = \frac{1}{2\pi I} \oint dz' \frac{f(z')}{z' - z} \tag{4.12}$$

is simply Stokes's theorem for the plane[6]. The bivector I^{-1} is necessary to rotate the line element dz' into the direction of the outward normal.

This definition (4.8) of an analytic function generalises easily to higher dimensions, where these functions are called *monogenic*, although the simple link with power series disappears. Again, there are some surprises in three dimensions. We have all learned about the important class of *harmonic* functions, defined as those functions $\psi(\mathbf{r})$ satisfying the scalar operator equation

$$\boldsymbol{\nabla}^2 \psi = 0. \tag{4.13}$$

Since monogenic functions satisfy

$$\boldsymbol{\nabla} \psi = 0, \tag{4.14}$$

they must also be harmonic. However, this first-order equation is more restrictive, so that not all harmonic functions are monogenic. In two dimensions, the solutions of equation (4.13) are written in terms of polar coordinates (r, θ) as

$$\psi = \left\{ \begin{array}{c} r^n \\ r^{-n} \end{array} \right\} \left\{ \begin{array}{c} \cos n\theta \\ \sin n\theta \end{array} \right\}. \tag{4.15}$$

Complex analysis tells us that there are special combinations (analytic functions) which have particular radial dependence:

$$\psi_1 = r^n (\cos n\theta + I \sin n\theta) = z^n, \tag{4.16}$$

$$\psi_2 = r^{-n} (\cos n\theta - I \sin n\theta) = z^{-n}. \tag{4.17}$$

In this way we can, in two dimensions, separate any given angular component into parts regular at the origin (r^n) and at infinity (r^{-n}). These parts are just the spinor solutions of the first-order equation (4.14).

The situation is exactly the same in three dimensions. The solutions of $\nabla^2\psi = 0$ are

$$\psi = \left\{ \begin{array}{c} r^l \\ r^{-l-1} \end{array} \right\} P_l^m(\cos\theta) e^{im\phi}, \tag{4.18}$$

but we can find specific combinations of angular dependence which are associated with a radial dependence of r^l or r^{-l-1}. We show this by example for the case $l = 1$. Obviously, non-trivial solutions of $\boldsymbol{\nabla}\psi = 0$ must contain more than just a scalar part — they must be multivectors. For the position vector $\mathbf{r}$ we find the following relations:

$$\begin{aligned} \boldsymbol{\nabla}\mathbf{r}\sigma_3 &= 3\sigma_3, & (4.19)\\ \boldsymbol{\nabla}\sigma_3\mathbf{r} &= -\sigma_3, & (4.20)\\ \boldsymbol{\nabla}r^k &= kr^{k-2}\mathbf{r}. & (4.21) \end{aligned}$$

(Equation 4.20 can be derived from a more general formula given in Section 5.) We can assemble solutions proportional to r and r^{-2}:

$$\begin{aligned} \frac{1}{2}(3\sigma_3\mathbf{r} + \mathbf{r}\sigma_3) &= r\,(2\cos\theta + i\sigma_\phi \sin\theta) & (4.22)\\ |r|^{-3}\mathbf{r}\sigma_3 &= r^{-2}(\cos\theta - i\sigma_\phi\sin\theta), & (4.23) \end{aligned}$$

where σ_ϕ is the unit vector in the azimuthal direction.

Alternatively, we can generate a spherical monogenic ψ from any spherical harmonic Φ:

$$\psi = \boldsymbol{\nabla}\Phi\sigma_3. \tag{4.24}$$

We have chosen to place the vector σ_3 to the right of Φ so as to keep ψ within the even subalgebra of spinors. This practice is also consistent with the conventional Pauli matrix representation (2.20)[24]. As an example, we try this procedure on the $l = 0$ harmonics:

$$\begin{array}{rcl} \Phi = 1 & \longrightarrow & \psi = 0 \\ \Phi = r^{-1} & \longrightarrow & \psi = r^{-2}(\cos\theta - i\sigma_\phi\sin\theta). \end{array} \tag{4.25}$$

For a selection of $l = 1$ harmonics we obtain

$$\begin{array}{rcl} \Phi = r\cos\theta & \longrightarrow & \psi = \sigma_3\sigma_3 = 1 \\ \Phi = r^{-2}\cos\theta & \longrightarrow & \psi = r^{-3}(-(3\cos^2\theta - 1) + 3i\sigma_\phi\sin\theta\cos\theta). \end{array} \tag{4.26}$$

Some readers may now recognise this process as similar to that in quantum mechanics when we add the spin contribution to the orbital angular momentum, making a total angular momentum $j = l \pm \frac{1}{2}$. The combinations of angular dependence are the same as in stationary solutions of the Dirac equation. In particular, (4.25) indicates that only one monogenic arises from $l = 0$. That is correct — only the $j = \frac{1}{2}$ state exists. Turning to (4.26) we see that there is one state with no angular dependence at all, and that the other has terms proportional to $P_2^m(\cos\theta)$. These can also be interpreted in terms of $j = \frac{1}{2}$ and $j = \frac{3}{2}$ respectively.

The process by which we have generated these functions has, of course, nothing to do with quantum mechanics — another clue that many quantum-mechanical procedures are much more classical than they seem.

5 The Algebra of Spacetime

The spacetime of Einstein's relativity is 4-dimensional, but with a difference. So far we have assumed that the square of any vector x is a scalar, and that $x^2 \geq 0$. For spacetime it is appropriate to make a different choice. We take the $(+ - - -)$ metric usually preferred by physicists, with a basis for the spacetime algebra[20] (STA) made up by the orthonormal vectors

$$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}, \quad \text{where} \quad \gamma_0^2 = -\gamma_k^2 = 1 \quad (k = 1, 2, 3). \tag{5.1}$$

These vectors $\{\gamma_\mu\}$ obey the same algebraic relations as Dirac's γ-matrices, but our interpretation of them is not that of conventional relativistic quantum mechanics. We do not view these objects as the components of a strange vector of matrices, but (as with the Pauli matrices of 3-space) as four separate vectors, with a clear geometric meaning.

From this basis set of vectors we construct the 16 $(= 2^4)$ geometric elements of the STA:

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\sigma_k, i\sigma_k\} & \{i\gamma_\mu\} & i \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 pseudovectors} & \text{1 pseudoscalar} \end{array} . \tag{5.2}$$

The time-like bivectors $\sigma_k \equiv \gamma_k\gamma_0$ are isomorphic to the basis vectors of 3-dimensional space; in the STA they represent an orthonormal frame of vectors in space *relative* to the laboratory time vector γ_0[5, 20]. The unit pseudoscalar of spacetime is defined as

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3, \tag{5.3}$$

which is indeed consistent with our earlier definition.

The geometric properties of spacetime are built into the mathematical language of the STA — it is the natural language of relativity. Equations written in the STA are *invariant* under passive coordinate transformations. For example, we can write the vector x in terms of its components $\{x^\mu\}$ as

$$x = x^\mu \gamma_\mu. \tag{5.4}$$

These components depend on the frame $\{\gamma_\mu\}$ and change under passive transformations, but the vector x is itself invariant. Conventional methods already make good use of scalar invariants in relativity, but much more power is available using the STA.

Active transformations are performed by rotors R, which are again even multivectors satisfying $R\tilde{R} = 1$:

$$e_\mu = R\gamma_\mu\tilde{R}, \tag{5.5}$$

where the $\{e_\mu\}$ comprise a new frame of orthogonal vectors. Any rotor R can be written as

$$R = \pm e^B, \tag{5.6}$$

where $B = \mathbf{a} + i\mathbf{b}$ is an arbitrary 6-component bivector ($\mathbf{a}$ and $\mathbf{b}$ are relative vectors). When performing rotations in higher dimensions, a simple rotation is defined by a *plane*, and cannot be characterised by a rotation axis; it is an accident of 3-dimensional space that planes can be mapped to lines by the duality operation. Geometric algebra brings this out clearly by expressing a rotation directly in terms of the plane in which it takes place.

For the 4-dimensional generalisation of the gradient operator ∇, we take account of the metric and write

$$\nabla \equiv \gamma^\mu \partial_\mu, \tag{5.7}$$

where the $\{\gamma^\mu\}$ are a *reciprocal frame* of vectors to the $\{\gamma_\mu\}$, defined via $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$.

As an example of the use of STA, we consider electromagnetism, writing the electromagnetic field in terms of the 4-potential A as

$$F = \nabla \wedge A = \nabla A - \nabla \cdot A. \tag{5.8}$$

The divergence term $\nabla \cdot A$ is zero in the Lorentz gauge. The field bivector F is expressed in terms of the more familiar electric and magnetic fields by making a space-time split in the γ_0 frame:

$$F = \mathbf{E} + i\mathbf{B}, \tag{5.9}$$

where

$$\mathbf{E} = \tfrac{1}{2}(F - \gamma_0 F \gamma_0), \quad i\mathbf{B} = \tfrac{1}{2}(F + \gamma_0 F \gamma_0). \tag{5.10}$$

Particularly striking is the fact that Maxwell's equations[20, 27] can be written in the simple form

$$\nabla F = J, \tag{5.11}$$

where J is the 4-current. Equation (5.11) contains *all* of Maxwell's equations because the ∇ operator is a vector and F is a bivector, so that the geometric product has both vector and trivector components. This trivector part is identically zero in the absence of magnetic charges. It is worth emphasising[5] that this compact formula (5.11) is not just a trick of notation, because the ∇ operator is invertible. We can, therefore, solve for F:

$$F = \nabla^{-1} J. \tag{5.12}$$

The inverse operator is known to physicists in the guise of the Green's propagators of relativistic quantum mechanics. We return to this point in a companion paper[16], in which we demonstrate this inversion explicitly for diffraction theory.

It is possible here, as in three dimensions, to represent a relativistic quantum-mechanical spinor (a Dirac spinor) by the *even* subalgebra of the STA[7, 24], which is 8-dimensional. We write this spinor as ψ and, since $\psi\tilde{\psi}$ contains only grade-0 and grade-4 terms, we decompose ψ as

$$\psi = \left(\rho e^{i\beta}\right)^{\frac{1}{2}} R, \tag{5.13}$$

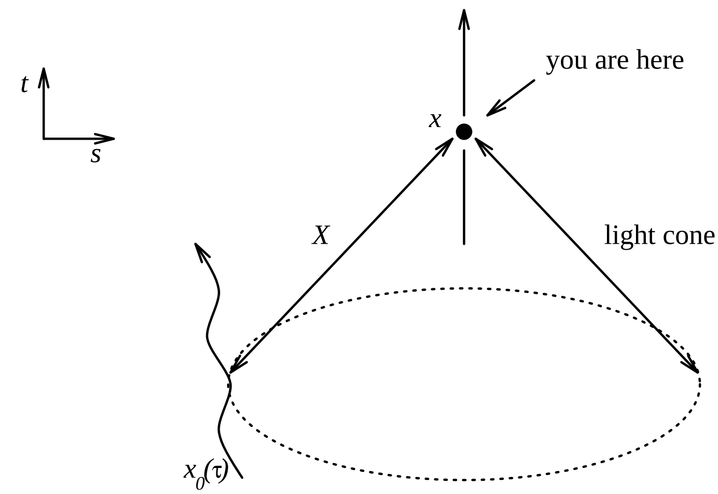

Figure 2: *A charge moving in the observer's past light-cone*

where R is a spacetime rotor. Thus, a relativistic spinor also contains an instruction to rotate — in this case to carry out a full Lorentz rotation. The monogenic equation in spacetime is simply

$$\nabla\psi = 0, \tag{5.14}$$

which, remarkably, is also the STA form of the massless Dirac equation[24]. Furthermore, the inclusion of a mass term requires only a simple modification:

$$\nabla\psi = m\psi\gamma_3 i. \tag{5.15}$$

As a final example of the power of the STA in relativistic physics, we give a compact formula for the fields of a radiating charge. This derivation is as explicit as possible, in order to give readers new to the STA some feeling for its character, but nevertheless it is still as compact as any of the conventional treatments in the literature. Let a charge q move along a world-line defined by $x_0(\tau)$, where τ is proper time. An observer at spacetime position x receives an electromagnetic influence from the charge when it lies on that observer's past light-cone (Figure 2). The vector

$$X \equiv x - x_0(\tau) \tag{5.16}$$

is the separation vector down the light-cone, joining the observer to this intersection point. We can take equation (5.16), augmented by the condition $X^2 = 0$, to define a mapping from the spacetime position x to a value of the particle's proper time τ. In this sense, we can write $\tau = \tau(x)$, and treat τ as a scalar field. If the charge is at rest in the observer's frame we have

$$x_0(\tau) = \tau\gamma_0 = (t - r)\gamma_0, \tag{5.17}$$

where r is the 3-space distance from the observer to the charge (taking $c = 1$). For this simple case the 4-potential A is a pure $1/r$ electrostatic field, which we can write as

$$A = \frac{q}{4\pi\epsilon_0}\frac{\gamma_0}{X\cdot\gamma_0}, \tag{5.18}$$

because $X\cdot\gamma_0 = t - (t - r) = r$. Generalising to an arbitrary velocity v for the charge, relative to the observer, gives

$$A = \frac{q}{4\pi\epsilon_0}\frac{v}{X\cdot v}, \tag{5.19}$$

which is a particularly compact and clear form for the Liénard-Wiechert potential.

We now wish to differentiate the potential to find the Faraday bivector. This will involve some general results concerning differentiation in the STA, which we now set up; for further useful results see Chapter 2 of Hestenes & Sobczyk[6]. Since the gradient operator is a vector we must take account of its commutation properties. Though it is evident that $\nabla x = 4$, we need also to deal with expressions such as $\overset{*}{\nabla}\, a\, \overset{*}{x}$, where a is a vector, and where the stars indicate that the ∇ operates only on x rather than a. The result[6] is found by anticommuting the x past the a to give $ax = 2x{\cdot}a - xa$, and then differentiating this. Generalized to a grade-r multivector A_r in an n-dimensional space, we have

$$\overset{*}{\nabla}\, A_r\, \overset{*}{x} = (-1)^r(n-2r)A_r. \tag{5.20}$$

Thus, in the example given above, $\overset{*}{\nabla}\, a\, \overset{*}{x} = -2a$. (See equation (4.20) for a 3-dimensional application of this result.)

We will also need to exploit the fact that the chain rule applies in the STA as in ordinary calculus, so that (for example)

$$\nabla x_0 = \nabla\tau\, v, \tag{5.21}$$

since x_0 is a function of τ alone, and $dx_0/d\tau \equiv \dot{x}_0 = v$ is the particle velocity. In equation (5.21) we use the convention that (in the absence of brackets or overstars) ∇ only operates on the object immediately to its right.

Armed with these results, we can now proceed quickly to the Faraday bivector. First, since

$$0 = \nabla X^2 = \nabla X X + \overset{*}{\nabla}\, X\, \overset{*}{X} = (4 - \nabla\tau\, v)X + (-2X - \nabla\tau\, Xv), \tag{5.22}$$

it follows that

$$\nabla\tau = \frac{X}{X{\cdot}v}. \tag{5.23}$$

As an aside, finding an explicit expression for $\nabla\tau$ confirms that the particle proper time can be treated as a scalar field — which is, perhaps, a surprising result. In the terminology of Wheeler & Feynman[28], such a function is called an 'adjunct field', because it obviously carries no energy or charge, being merely a mathematical device for encoding information. We share the hope of Wheeler & Feynman that some of the paradoxes of classical and quantum electrodynamics, in particular the infinite self-energy of a point charge, might be avoidable by working with adjunct fields of this kind.

To differentiate A, we need $\nabla(X{\cdot}v)$. Using the results already established we have

$$\nabla(vX) = \nabla\tau\, \dot{v}X - 2v - \nabla\tau\, v^2 = \frac{X\dot{v}X - 2X - vXv}{X{\cdot}v}, \tag{5.24}$$

$$\nabla(Xv) = \nabla\tau\, X\dot{v} + 4v - \nabla\tau\, v^2 = \frac{2vXv + X}{X{\cdot}v}, \tag{5.25}$$

which combine to give

$$\nabla(X{\cdot}v) = \frac{X\dot{v}X - X + vXv}{2(X{\cdot}v)}. \tag{5.26}$$

This yields

$$\nabla A = \frac{q}{4\pi\epsilon_0}\left\{\frac{\nabla v}{X{\cdot}v} - \frac{1}{(X{\cdot}v)^2}\nabla(X{\cdot}v)v\right\}$$

$$= \frac{q}{8\pi\epsilon_0 (X\cdot v)^3}\left(X\dot{v}vX + Xv - vX\right), \tag{5.27}$$

so that

$$\nabla\cdot A = 0 \tag{5.28}$$

and

$$F = \frac{q}{4\pi\epsilon_0}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}. \tag{5.29}$$

Here, Ω_v is the 'acceleration bivector' of the particle:

$$\Omega_v \equiv \dot{v}v. \tag{5.30}$$

The quantity $\dot{v}v = \dot{v}\wedge v$ is a pure bivector, because $v^2 = 1$ implies that $\dot{v}\cdot v = 0$. For more on the value of representing the acceleration in terms of a bivector, and the sense in which Ω_v is the rest-frame component of a more general acceleration bivector, see Chapter 6 of Hestenes & Sobczyk[6].

The form of the Faraday bivector given by equation (5.29) is very revealing. It displays a clean split into a velocity term proportional to $1/(\text{distance})^2$ and a long-range radiation term proportional to $1/(\text{distance})$. The first term is exactly the Coulomb field in the rest frame of the charge, and the radiation term,

$$F_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\frac{1}{2}X\Omega_v X}{(X\cdot v)^3}, \tag{5.31}$$

is proportional to the rest-frame acceleration projected down the null-vector X.

Finally, we return to the subject of adjunct fields. Clearly X is an adjunct field, as $\tau(x)$ was. It is easy to show that

$$A = \frac{-q}{8\pi\epsilon_0}\nabla^2 X, \tag{5.32}$$

so that

$$F = \frac{-q}{8\pi\epsilon_0}\nabla^3 X. \tag{5.33}$$

In this expression for F we have expressed a physical field solely in terms of a derivative of an 'information carrying' adjunct field. Expressions such as (5.32) and (5.33) (which we believe are new, and were derived independently by ourselves and David Hestenes) may be of further interest in the elaboration of Wheeler-Feynman type 'action at a distance' ideas[28, 29].

6 Concluding Remarks

Most of the above is well known — the vast majority of the theorems presented date back at least a hundred years. The trouble, of course, is that these facts, whilst 'known', were not known by the right people at the right time, so that an appalling amount of reinvention and duplication took place as physics and mathematics advanced. Spinors have a central role in our understanding of the algebra of space, and they have accordingly been reinvented more often than anything else. Each reincarnation emphasises different aspects and uses different notation, adding new storeys to our mathematical Tower of Babel. What we have tried to show in this introductory paper is that the geometric algebra of David Hestenes provides the best framework by which to unify these disparate approaches. It is our earnest hope that more physicists will come to use it as the main language for expressing their work.

In the three following papers, we explore different aspects of this unification, and some of the new physics and new insights which geometric algebra brings. Paper II[24] discusses the translation into geometric algebra of other languages for describing spinors and quantum-mechanical states and operators, especially in the context of the Dirac theory. It will be seen that Hestenes' form of the Dirac equation genuinely liberates it from any dependence upon specific matrix representations, making its intrinsic geometric content much clearer.

Paper III[27] uses the concept of multivector differentiation[6] to make many unifications and improvements in the area of Lagrangian field theory. The use of a consistent and mathematically rigorous set of tools for spinor, vector and tensor fields enables us to clarify the role of antisymmetric terms in stress-energy tensors, about which there has been some confusion. A highlight is the inclusion of functional differentiation within the framework of geometric algebra, enabling us to treat 'differentiation with respect to the metric' in a new way. This technique is commonly used in field theories as one means of deriving the stress-energy tensor, and our approach again clarifies the role of antisymmetric terms.

Paper IV[16] examines in detail the physical implications of Hestenes' formulation and interpretation of the Dirac theory. New results include predictions for the time taken for an electron to traverse the classically-forbidden region of a potential barrier. This is a problem of considerable interest in the area of semiconductor technology.

We have shown elsewhere how to translate Grassmann calculus[30, 31], and some aspects of twistor theory[32] into geometric algebra, with many simplifications and fresh insights. Thus, geometric algebra spans very large areas of both theoretical and applied physics.

There is another language which has some claim to achieve useful unifications. The use of 'differential forms' became popular with physicists, particularly as a result of its use in the excellent, and deservedly influential, 'Big Black Book' by Misner, Thorne & Wheeler[33]. Differential forms are skew multilinear functions, so that, like multivectors of grade k, they achieve the aim of coordinate independence. By being scalar-valued, however, differential forms of different grades cannot be combined in the way multivectors can in geometric algebra. Consequently, rotors and spinors cannot be so easily expressed in the language of differential forms. In addition, the 'inner product', which is necessary to a great deal of physics, has to be grafted into this approach through the use of the duality operation, and so the language of differential forms never unifies the inner and outer products in the manner achieved by geometric algebra.

This leads us to say a few words about the widely-held opinion that, because complex numbers are fundamental to quantum mechanics, it is desirable to 'complexify' every bit of physics, including spacetime itself. It will be apparent that we disagree with this view, and hope earnestly that it is quite wrong, and that complex numbers (as mystical uninterpreted scalars) will prove to be unnecessary even in quantum mechanics.

The same sentiments apply to theories involving spaces with large numbers of dimensions that we do not observe. We have no objection to the use of higher dimensions as such; it just seems to us to be unnecessary at present, when the algebra of the space that we *do* observe contains so many wonders that are not yet generally appreciated.

We leave the last words to David Hestenes and Garret Sobczyk[6]:

Geometry without algebra is dumb! — Algebra without geometry is blind!

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