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Information in the Zero Crossings of Bandpass Signals

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An interesting subclass of bandpass signals $\{h\}$ is described wherein the zero crossings of h determine h within a multiplicative constant. The members may have complex zeros, but it is necessary that h should have no zeros in common with its Hilbert transform $\hat{h}$ other than real simple zeros. It is then sufficient that the band be less than an octave in width. The subclass is shown to include full-carrier upper-sideband signals (of less than an octave bandwidth). Also it is shown that full-carrier lower-sideband signals have only real simple zeros (for any ratio of upper and lower frequencies) and, hence, are readily identified by their zero crossings. However, under the most general conditions for uniqueness, the problem of actually recovering h from its sign changes appears to be very difficult and impractical.

I. INTRODUCTION

Voelcker and Requicha[1] raised the question, among others, as to when a bandpass signal $h(t)$ might be recovered (within a multiplicative constant) from sgn $\{h(t)\}$, that is, from its zero crossings. There are really two questions here that should be treated separately: the question of uniqueness and the question of recoverability. Recoverability implies that there is an effective (stable) way of recovering the signal from the data. Uniqueness does not always imply recoverability. For example, a band-limited signal is uniquely determined from its samples (at slightly greater than the Nyquist rate) just on a half line, say $t < 0$, but there is no stable way of recovering the signal from the half-line samples. However, to demonstrate recoverability we must first establish uniqueness. Here we examine the question of uniqueness.

There are countless ad hoc ways of choosing a subset Z of bandpass signals such that

$$\operatorname{sgn}\{h_1(t)\} \equiv \operatorname{sgn}\{h_2(t)\}, \quad h_1 \in Z, \quad h_2 \in Z \tag{1}$$

implies

$$h_1(t) \equiv A h_2(t), \tag{2}$$

e.g., by choosing the first member in an arbitrary way and then choosing successive members that have distinct signum functions. However, the subset Z could be considered interesting only if it reveals basic constraints on the sign changes of members of the whole class. Our objective is to illuminate the structure of bandpass functions (signals) having the same signum function.

In connection with (1), we are going to assume that the function $\operatorname{sgn}\{h(t)\}$ has no removable discontinuities,* and, hence, does not mark the location of zeros of even multiplicity. Also, in the context of the problem here we say two functions are *distinct* only if one is not a constant multiple of the other.

We first focus on the problem of constructing distinct bandpass functions having the same signum function. This leads to the concept of the "free" zeros of a bandpass function h.

The free zeros of h are those zeros that may be removed or moved around (by replacing the removed zero with another) without destroying the bandpass property of h. Removing (or moving) any zero of h does not destroy the overall low-pass property of h but may destroy the bandpass property. The simple examples $\sin t$ and $t^{-1} \sin t$ illustrate this fact.

We show that the free zeros of h are simply the common zeros of h and its Hilbert transform $\hat{h}$. These are further identified as common zeros of certain low-pass functions in the representation of h. In case of real-valued $h(t)$, the free zeros of h are conveniently identified in the representation

$$h(t) = \operatorname{Re}\{f(t)e^{i\mu t}\}$$

as the real zeros of f and those complex zeros of f that occur in conjugate

* According to the usual convention, $\operatorname{sgn} 0 = 0$, the function $s(t) = \operatorname{sgn}\{\sin^2 t\}$ would have removable discontinuities at the zeros of $\sin t$. Here we assume that the function of t, $\operatorname{sgn}\{h(t)\}$, takes the value 0 only at points where $h(t)$ changes sign.

pairs. Here $f(t)$ is an arbitrary *complex-valued* band-limited function which need not have complex zeros occurring in conjugate pairs.

It follows readily that if a real-valued bandpass $h(t)$ has free zeros other than real simple (free) zeros, then there is a distinct function in the same class having the same signum function as $h(t)$. So, in the absence of some meaningless ad hoc rule, we must restrict our attention to functions that have no free zeros other than real simple zeros if we require sgn $\{h(t)\}$ to determine $h(t)$ within a constant multiplier.

It is possible, however, as shown by an example, for *distinct* bandpass functions to have the same signum function when *neither* has any free zeros. This is possible only when the passband spans an octave or more.

Our main result is that (1) implies (2) when $Z = Z(\alpha,\beta)$ consists of those real-valued $h(t)$ having no free zeros other than real simple free zeros and having spectrum confined to $[\alpha,\beta]$ (and $[-\beta,-\alpha]$), where $0 < \alpha < \beta < 2\alpha$.

The key to this result is the simple identity (37)

$$\hat{h}_1(t)h_2(t) - \hat{h}_2(t)h_1(t) = g(t),$$

where in terms of the representation (9),

$$h_i(t) = p_i(t)\cos\mu t - q_i(t)\sin\mu t \quad (i = 1,2),$$

g is given by

$$g(t) = p_2(t)q_1(t) - p_1(t)q_2(t).$$

Here p_i and q_i are band limited to $[-\lambda/2,\lambda/2]$ and, hence, g is band limited to $[-\lambda,\lambda]$, where $\lambda = \beta - \alpha$. Then, if h_1 and h_2 have enough common zeros $\{t_k\} = S$, we can conclude from

$$g(t_k) = 0 \quad \text{all } t_k \text{ in } S$$

that

$$g(t) \equiv 0,$$

and, hence, that

$$\frac{h_1(t)}{\hat{h}_1(t)} \equiv \frac{h_2(t)}{\hat{h}_2(t)}.$$

Then, if h_1 and h_2 have *no* free zeros, i.e., *no* zeros in common with their Hilbert transforms, we can conclude that

$$h_1(t) \equiv Ah_2(t).$$

The same conclusion can be obtained with some additional argument when

$$h_1(t_k) = h_2(t_k) = 0, \quad \text{all } t_k \text{ in } S$$

is replaced by

$$\operatorname{sgn} h_1(t) \equiv \operatorname{sgn} h_2(t)$$

and h_1 and h_2 are allowed free zeros that are only real and simple.

It is well known that g can have no more zeros, roughly speaking, than $\cos \lambda t$ without vanishing identically. It is also known that h_i must have, roughly speaking, at least as many sign changes as $\cos \alpha t$. In any particular case, all we really need in addition to the free-zero constraint is that h_i has, roughly speaking, more sign changes than $\cos \lambda t$, where λ is the width of the passband. This is always assured, then, when $\alpha > \lambda$, i.e., when $\beta < 2\alpha$, but of course may obtain in other cases.

For the rigorous development of our results we first require some basic definitions.

II. BAND-LIMITED FUNCTIONS

These are restrictions to the real line of entire functions of exponential type, which are bounded on the real line. The standard reference on the subject is the book by Boas.[2] It is convenient to introduce a notation for subclasses of band-limited functions.

Definition: $B_p(\lambda)$, $(1 \leqq p \leqq \infty)$ denotes the collection of functions $f(t)$, $-\infty < t < \infty$, which belong to L_p on the real line and extend as entire functions $f(\tau)$, $\tau = t + iu$, of exponential type λ, $\lambda \geqq 0$. [$B_p(0)$ is empty except for $p = \infty$.]

For $1 \leqq p \leqq 2$, the functions in $B_p(\lambda)$ have ordinary Fourier transforms that vanish outside $[-\lambda,\lambda]$. This follows from the Paley-Wiener theorem[3] for B_2 and the fact that*

$$B_p(\lambda) \subset B_{p'}(\lambda), \quad p' > p. \tag{3}$$

Thus, $B_\infty(\lambda)$ contains $B_p(\lambda)$ for all $p \geqq 1$ and it has been shown[4] that for f in $B_\infty(\lambda)$ (see Appendix),

$$\lim_{T \to \infty} \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) f(t) e^{-i\omega t} dt = 0, \quad |\omega| > \lambda. \tag{4}$$

So, in a very real sense, the Fourier transforms of functions in $B_\infty(\lambda)$ can be said to vanish outside $[-\lambda,\lambda]$.

III. BANDPASS FUNCTIONS

These are bounded functions whose spectra are confined to the intervals $[\alpha,\beta]$ and $[-\beta,-\alpha]$ where $0 < \alpha < \beta < \infty$.

* See Ref. 2, Theorem 6.7.18, page 102.

Definition: $B_p(\alpha,\beta)$ denotes the class of functions of the form

$$h(t) = f_1(t)e^{i\mu t} + f_2(t)e^{-i\mu t}, \tag{5}$$

where f_1 and f_2 belong to $B_p(\lambda/2)$,

$$\lambda = \beta - \alpha \quad (0 < \alpha < \beta < \infty) \tag{5a}$$

$$\mu = \frac{\alpha + \beta}{2}. \tag{5b}$$

It follows from (3) that

$$B_\infty(\alpha,\beta) \supset B_p(\alpha,\beta), \quad 1 \leqq p < \infty,$$

so we focus on the more general class $B_\infty(\alpha,\beta)$.

Functions of the form (5) have Hilbert transforms (see Ref. 5) $\hat{h}(t)$ given by

$$\hat{h}(t) = -if_1(t)e^{i\mu t} + if_2(t)e^{-i\mu t}. \tag{6}$$

(We could take (6) as the definition of the Hilbert transform of a bounded bandpass function and show that it agrees with the usual definition.) We have

$$h(t) + i\hat{h}(t) = 2f_1(t)e^{i\mu t} \tag{7}$$

$$h(t) - i\hat{h}(t) = 2f_2(t)e^{-i\mu t} \tag{8}$$

We may write (5) and (6) in the forms

$$h(t) = p(t)\cos\mu t - q(t)\sin\mu t \tag{9}$$

$$\hat{h}(t) = p(t)\sin\mu t + q(t)\cos\mu t, \tag{10}$$

where

$$p(t) = f_1(t) + f_2(t); \quad q(t) = if_2(t) - if_1(t). \tag{11}$$

Then for real-valued $h(t)$, we must have p and q real and, therefore,

$$f_2(t) = \overline{f_1(t)}. \tag{12}$$

That is, a real-valued function in $B_\infty(\alpha,\beta)$ is completely described by one complex-valued function f in $B_\infty(\lambda/2)$, $\lambda = \beta - \alpha$, or equivalently by two real-valued functions p and q in $B_\infty(\lambda/2)$; i.e.,

$$h(t) = \operatorname{Re}\{f(t)e^{i\mu t}\}, \tag{13}$$

where

$$f(t) = p(t) + iq(t), \qquad p,q \in B_\infty(\lambda/2). \tag{13a}$$

It is sometimes convenient to exhibit one of the end points of the interval $[\alpha,\beta]$ in the exponential factor by writing for (13)

$$h(t) = \operatorname{Re}\{f_+(t)e^{i\alpha t}\}, \tag{14}$$

where

$$f_+(t) = f(t) \exp\left\{i \frac{\lambda t}{2}\right\} \equiv x(t) + iy(t) \tag{14a}$$

or

$$h(t) = \text{Re}\,\{f_-(t)e^{i\beta t}\}, \tag{15}$$

where

$$f_-(t) = f(t) \exp\left\{-\frac{i\lambda t}{2}\right\} \equiv r(t) - is(t). \tag{15a}$$

In (14a) f_+ is a function whose spectrum is confined to $[0,\lambda]$ and whose real and imaginary parts x and y belong to $B_\infty(\lambda)$. In (15a) f_- is a function whose spectrum is confined to $[-\lambda,0]$ and whose real and imaginary parts r and $-s$ belong to $B_\infty(\lambda)$. In the form (14), $h(t)$ may be interpreted as the upper single-sideband signal associated with $x(t)$ and carrier frequency α, whereas in (15), $h(t)$ may be interpreted as the lower single-sideband signal associated with $r(t)$ and carrier frequency β. Usually one thinks of y as the Hilbert transform of x and s as the Hilbert transform of r. However, such a relation does not follow without further restrictions on f; e.g., $f \in B_p(\lambda/2)$, $p < \infty$. Because x and y (r and s) are interdependent through p and q, the representation (13) is usually more convenient to work with.

IV. FREE ZEROS OF BANDPASS FUNCTIONS

If h belongs to $B_\infty(\alpha,\beta)$ and $h(\xi) = 0$, then the function

$$g(t) \equiv \frac{at + b}{t - \xi} h(t) \tag{16}$$

for arbitrary (a,b) certainly belongs to $B_\infty(\beta)$, since $g(\tau)$ is an entire function of exponential type β bounded on the real line. However, it does not follow that g belongs to $B_\infty(\alpha,\beta)$. For this reason, it is not so easy to construct distinct bandpass functions having the same signum function.

Definition: A complex (or real) number ξ is said to be a free zero of h if the function g defined in (16) belongs to $B_\infty(\alpha,\beta)$ whenever h belongs to $B_\infty(\alpha,\beta)$.

Theorem 1: *A complex (or real) number ξ is a free zero of a function h in $B_\infty(\alpha,\beta)$ if and only if*

$$h(\xi) = 0$$

and

$$\hat{h}(\xi) = 0.$$

(In other words, the free zeros of h are the common zeros of h and its Hilbert transform).

Proof: In order for the function g in (16) to belong to $B_\infty(\alpha,\beta)$, it must be of the form (5); i.e.,

$$g(t) = g_1(t)e^{i\mu t} + g_2(t)e^{-i\mu t} \tag{17}$$
$$g_1, g_2 \quad \text{in } B_\infty(\lambda/2).$$

And since

$$h(t) = f_1(t)e^{i\mu t} + f_2(t)e^{-i\mu t} \tag{18}$$
$$f_1, f_2 \quad \text{in } B_\infty(\lambda/2),$$

we must have

$$g_1(t) \equiv \frac{at+b}{t-\xi} f_1(t) \quad \text{in } B_\infty(\lambda/2) \tag{19}$$

$$g_2(t) \equiv \frac{at+b}{t-\xi} f_2(t) \quad \text{in } B_\infty(\lambda/2) \tag{20}$$

and, therefore, we must have

$$f_1(\xi) = 0 \tag{21}$$

$$f_2(\xi) = 0; \tag{22}$$

and, hence, from (6)

$$\hat{h}(\xi) = 0. \tag{23}$$

So $h(\xi) = \hat{h}(\xi) = 0$ is a necessary condition for ξ to be a free zero. On the other hand, if

$$h(\xi) = f_1(\xi)e^{i\mu\xi} + f_2(\xi)e^{-i\mu\xi} = 0 \tag{24}$$

and

$$\hat{h}(\xi) = -if_1(\xi)e^{i\mu\xi} + if_2(\xi)e^{-i\mu\xi} = 0, \tag{25}$$

it follows that

$$f_1(\xi) = f_2(\xi) = 0, \tag{26}$$

and, hence, that g_1 and g_2 defined in (19) and (20) belong to $B_\infty(\lambda/2)$ and, therefore, that g defined in (16) belongs to $B_\infty(\alpha,\beta)$. Hence, $h(\xi) = \hat{h}(\xi) = 0$ is a necessary and sufficient condition for ξ to be a free zero of h. In the course of the proof, we have established the following results which we label for future reference.

Theorem 2: If h belongs to $B_\infty(\alpha,\beta)$ and $h(\xi) = \hat{h}(\xi) = 0$, then $g(t) = (at + b)/(t - \xi)h(t)$ belongs to $B_\infty(\alpha,\beta)$ and has the Hilbert transform

$$\hat{g}(t) = \frac{at+b}{t-\xi}\hat{h}(t) \quad \textit{also in } B_\infty(\alpha,\beta).$$

Theorem 3: *The free zeros (if any) of a function of the form (5) are the common zeros of f_1 and f_2, or equivalently, the common zeros of p and q in the representation (9).*

Corollary 3.1: *A real-valued function h of the form (13) has a free zero ξ if and only if*

$$f(\xi) = f(\bar{\xi}) = 0.$$

Corollary 3.2: *If the function $f(\tau)$, $\tau = t + iu$, in (13) is zero-free in either half-plane $u \geqq 0$ or $u \leqq 0$ then h has no free zeros.*

In connection with Corollary 3.2, we note that for f to be zero-free in the (closed) upper half-plane $u \geqq 0$, it is sufficient that $x(t)$ defined in (14a) satisfy

$$x(t) > 0, \quad -\infty < t < \infty. \tag{27}$$

Also, for f to be zero-free in the (closed) lower half-plane $u \leqq 0$, it is sufficient that $r(t)$ defined in (15a) satisfy

$$r(t) > 0, \quad -\infty < t < \infty. \tag{28}$$

This follows from the fact that a function $f_+(\tau)$ bounded and analytic in the upper half-plane may be represented by the Poisson integral[6]

$$f_+(t + iu) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u}{(t - \xi)^2 + u^2} f_+(\xi) d\xi \quad u > 0.$$

Hence, if $\mathrm{Re}\,\{f_+(t)\} = x(t) > 0$, then $\mathrm{Re}\,\{f_+(t + iu)\} > 0$ for $u > 0$. A similar statement holds for functions $f_-(\tau)$ bounded and analytic in the lower half-plane.

Now the role of free zeros in the problem under consideration is made clear by the following:

Theorem 4: *If h_1 is a real-valued function in $B_\infty(\alpha,\beta)$ having a complex free zero $\xi = a + ib$, $b > 0$, or a multiple real free zero ξ, then there is a function h_2 in $B_\infty(\alpha,\beta)$ such that*

$$\mathrm{sgn}\,\{h_1(t)\} = \mathrm{sgn}\,\{h_2(t)\}, \quad -\infty < t < \infty$$

and

$$h_2(t) \neq Ah_1(t), \quad -\infty < t < \infty.$$

Proof: It follows from Corollary 3.1 that if $\xi = a + ib$, $b > 0$, is a free zero of h_1, then $\bar{\xi} = a - ib$ is also a free zero of h_1. Hence, we may take h_2 to be

$$h_2(t) = \frac{P_2(t) h_1(t)}{(t - \xi)(t - \bar{\xi})} \quad \text{in } B_\infty(\alpha,\beta), \tag{29}$$

where $P_2(t)$ is any polynomial of degree 2 satisfying

$$P_2(t) > 0, \quad -\infty < t < \infty, \quad P_2(t) \neq A(t-\xi)(t-\bar{\xi}). \tag{30}$$

In case ξ is a multiple real free zero of h_1 (i.e., of multiplicity $\geqq 2$), then (29) is still valid with $\xi = \bar{\xi}$.

The converse of Theorem 4 is not true. We need a condition on how often h_1 and h_2 vanish together.

V. BANDPASS FUNCTIONS WHICH VANISH TOGETHER ON LARGE SETS

Here we would like to investigate the implications of

$$h_1(\tau_k) = h_2(\tau_k) = 0, \quad \text{all } \tau_k \in S, \tag{31}$$

where h_1, h_2 belong to $B_\infty(\alpha,\beta)$ and S is a set of uniqueness for $B_\infty(\lambda)$, $\lambda = \beta - \alpha$. We suppose that (31) does not imply that h_1 or h_2 vanish identically.

Definition: $S = \{\tau_k\}$ is said to be a set of uniqueness for $B_\infty(\lambda)$ if

$$g(t) \quad \text{in } B_\infty(\lambda)$$

and

$$g(\tau_k) = 0 \quad \text{all } \tau_k \in S$$

imply

$$g(t) \equiv 0.$$

We do not assume that h_1 and h_2 are real-valued (on the real axis) and write, using (7), (8), and (11),

$$h_1(t) \pm i\hat{h}_1(t) = \{p_1(t) \pm iq_1(t)\}e^{\pm i\mu t} \tag{32}$$

$$h_2(t) \pm i\hat{h}_2(t) = \{p_2(t) \pm iq_2(t)\}e^{\pm i\mu t}, \tag{33}$$

where p_1, q_1, p_2, q_2 are arbitrary functions in $B_\infty(\lambda/2)$ and $\mu = (\alpha + \beta)/2 > \lambda/2$. Then,

$$\{h_1(t) + i\hat{h}_1(t)\}\{h_2(t) - i\hat{h}_2(t)\} = \{p_1(t) + iq_1(t)\}\{p_2(t) - iq_2(t)\} \tag{34}$$

$$\{h_1(t) - i\hat{h}_1(t)\}\{h_2(t) + i\hat{h}_2(t)\} = \{p_1(t) - iq_1(t)\}\{p_2(t) + iq_2(t)\}. \tag{35}$$

It follows from (34) and (35) that

$$h_1(t)h_2(t) + \hat{h}_1(t)\hat{h}_2(t) = p_1(t)p_2(t) + q_1(t)q_2(t) \in B_\infty(\lambda) \tag{36}$$

$$\hat{h}_1(t)h_2(t) - \hat{h}_2(t)h_1(t) = q_1(t)p_2(t) - p_1(t)q_2(t) \in B_\infty(\lambda). \tag{37}$$

Thus, the functions on the left in (36) and (37), apparently of type $\leqq 2\beta$, in fact are of type $\leqq \lambda$ as the functions on the right show. Then, from (37), if $h_1(\tau)$ and $h_2(\tau)$ vanish together on a set of uniqueness for $B_\infty(\lambda)$, the functions on the right and left vanish identically. We state this result as a theorem for future reference, using the representation (9).

Theorem 5: Let h_1 and h_2 belong to $B_\infty(\alpha,\beta)$

$$h_1(t) = p_1(t)\cos\mu t - q_1(t)\sin\mu t$$

$$h_2(t) = p_2(t)\cos\mu t - q_2(t)\sin\mu t$$

and

$$h_1(t) \not\equiv 0, \quad h_2(t) \not\equiv 0.$$

Then (31) implies

$$\frac{h_1(t)}{\hat{h}_1(t)} \equiv \frac{h_2(t)}{\hat{h}_2(t)} \equiv M(t) \tag{38}$$

and

$$p_1(t)q_2(t) \equiv q_1(t)p_2(t), \tag{39}$$

and, hence, if $q_1q_2 \not\equiv 0$,

$$\frac{p_1(t)}{q_1(t)} \equiv \frac{p_2(t)}{q_2(t)} \equiv N(t). \tag{40}$$

We should note in connection with (40) that $q_1 \equiv 0$ implies $p_1 \not\equiv 0$ (since $h_1 \not\equiv 0$) and, hence, from (39) that $q_2 \equiv 0$. By symmetry, $q_1q_2 \equiv 0$ implies

$$\begin{cases} h_1(t) = p_1(t)\cos\mu t, & h_2(t) = p_2(t)\cos\mu t \\ \hat{h}_1(t) = p_1(t)\sin\mu t, & \hat{h}_2(t) = p_2(t)\sin\mu t. \end{cases} \tag{41}$$

We cannot, according to the hypotheses, have $q_1 \equiv 0$ and $p_2 \equiv 0$ (or $p_1 \equiv 0$, $q_2 \equiv 0$), i.e.,

$$h_1(t) = p_1(t)\cos\mu t$$

$$h_2(t) = q_2(t)\sin\mu t$$

for then common zeros of h_1 and h_2 are necessarily common zeros of p_1 and q_2 so that (31) would imply, since p_1 and q_2 belong to $B_\infty(\lambda/2)$, that $h_1 \equiv 0$, $h_2 \equiv 0$, contrary to hypothesis. For a similar reason, $q_1 \equiv 0$ (or $q_2 \equiv 0$) implies that the set S in (31) includes a lot of the zeros of $\cos\mu t$ in (41).

Now the function $M(t)$ in (39) is a meromorphic function, the quotient of two functions in $B_\infty(\alpha,\beta) \subset B_\infty(\beta)$. The zeros of $M(t)$ are zeros of h_1 not common to $\hat{h}_1$. Hence, if h_1 and $\hat{h}_1$ have no common zeros, i.e., if h_1 has no free zeros, then the zeros of $M(t)$ are precisely the zeros of h_1. The zeros of a band-limited function determine the function within an exponential factor which in turn depends on the (actual) spectral end points. It follows from a theorem of Titchmarsh[7] (with an additional minor argument) that the zeros of a function f in $B_\infty(\beta)$ whose spectral end points are *centered* about the origin, i.e., a function whose spectrum is confined to $[-\beta',\beta']$, $(\beta' \leqq \beta)$ but to no smaller interval, determine the function within a constant multiplier. For such functions,

$$f(t) = At^m \prod_{k=1}^{\infty} \left(1 - \frac{t}{\tau_k}\right), \tag{42}$$

where

$$|\tau_{k+1}| \geqq |\tau_k| > 0$$

with the product converging conditionally (owing to the ordering of the zeros). In particular, (42) holds for a band-limited function which is real-valued on the real axis. Hence, if h_1 in Theorem 5 is real-valued on the real axis and has no free zeros, the zeros of $M(t)$ determine h_1 within a multiplicative constant. $M(t)$ is, in principle at least, determined by any non-null function in $B_\infty(\alpha,\beta)$, say h_2, which vanishes on S. There is by hypothesis at least one such function. We may state this result as follows:

Theorem 6: Let h_1 and h_2 belong to $B_\infty(\alpha,\beta)$ and be real-valued on the real axis and have no free zeros. Then,

$$h_1(\tau_k) = h_2(\tau_k) \quad \text{for all } \tau_k \text{ in } S,$$

where S is a set of uniqueness for $B_\infty(\lambda)$, $\lambda = \beta - \alpha$, implies (if $h_2 \not\equiv 0$)

$$h_1(t) \equiv Ah_2(t).$$

Actually, for the problem at hand, we are interested in sets S which consist of points $\{t_k\}$, where h_1 and h_2 change sign. If this set has an upper density in excess of λ/π, then it is well known (see Levinson[8] for example) that S is a set of uniqueness for $B_\infty(\lambda)$. So in Theorem 6 we may take S to be any set $\{t_k\}$ where the number $\nu(T)$ of t_k in the interval $(0,T)$ satisfies

$$\limsup_{T\to\infty} \frac{\nu(T)}{T} > \lambda/\pi. \tag{43}$$

Roughly speaking, if h_1 and h_2 just vanish together (not necessarily change sign together) more often than $\cos \lambda t$, then the conclusion follows. We know, furthermore, that real-valued functions in $B_\infty(\alpha,\beta)$ must change sign (on either half line), again roughly speaking, at least as often as $\cos \alpha t$.

Theorem 7 (from Ref. 9): Let h be a real-valued function in $B_\infty(\alpha,\beta)$, $h \neq 0$, and denote by $\sigma(T)$ the number of sign changes of $h(t)$ in the interval $(0,T)$. Then,

$$\liminf_{T\to\infty} \frac{\sigma(T)}{T} \geqq \alpha/\pi.$$

Hence, we have (since lim sup $\geqq$ lim inf):

Theorem 8: In Theorem 6 if $h_1 \not\equiv 0$, we may always take S to be the set $\{t_k\}$ where $h_1(t)$ changes sign provided $\alpha > \lambda$; i.e., provided $\alpha > \beta/2$.

The necessity of the strict inequality $\alpha > \beta/2$ in Theorem 8 is shown by the one-parameter family

$$h(t;a) = \text{Re}\,\{(1 + iae^{it})e^{it}\}, \quad -\tfrac{1}{2} < a < \tfrac{1}{2}$$
$$= \cos t - a \sin 2t = (1 - 2a \sin t) \cos t. \quad (44)$$

Here, $h(t;a)$ belongs to $B_\infty(1,2)$ and

$$\text{sgn}\,\{h(t;a)\} = \text{sgn}\,\{\cos t\}, \quad -\tfrac{1}{2} < a < \tfrac{1}{2}. \quad (45)$$

Also, $h(t;a)$ has no free zeros, which follows by identifying f in (13) as

$$f(t) = (1 + iae^{it})e^{-it/2},$$

which is clearly zero-free in the closed upper half-plane and, hence, by Corollary 3.2, $h(t;a)$ has no free zeros. Yet all members of the family have the same sign. There are similar examples for $B_\infty(m,n)$, m and n positive integers, $m < 2n$.

If $h_1(t)$ changes sign at $\{t_k\}$, then

$$\text{sgn}\,\{h_1(t)\} \equiv \text{sgn}\,\{h_2(t)\}$$

is a stronger statement than

$$h_1(t_k) = h_2(t_k) = 0.$$

By replacing the latter condition by the former, we can with a little more work obtain the conclusion of Theorem 6 by allowing h_1 and h_2 to have only real, simple, free zeros. (Note that $h(t)$ may have a high-order zero, say at $t = 0$, and yet have only a simple free zero there that would require only that $\hat{h}(t)$ have a simple zero at $t = 0$.) This is the most we could hope for in view of Theorem 4 and the example in eq. (44).

We denote by $Z(\alpha,\beta)$ the class of (real) bandpass functions that have no free zeros other than simple, real, free zeros. That is,

Definition: $Z(\alpha,\beta)$, $0 < \alpha < \beta < \infty$, consists of all real-valued functions $h(t)$ of the form

$$h(t) = \text{Re}\,\{f(t)e^{i\mu t}\},$$

where $\mu = (\alpha + \beta)/2$ and $f(t)$ belongs $B_\infty(\lambda/2)$, $\lambda = \beta - \alpha$, and has no pair of complex conjugate zeros and no real zeros that are not simple.

We should note that $Z(\alpha,\beta)$ includes all real-valued functions in $B_\infty(\alpha,\beta)$ that have only real simple zeros. For if h_0 is such a function, then f_0 in the above representation has no pair of complex conjugate zeros, since these are common zeros of h_0 and $\hat{h}_0$. Similarly, f_0 can have no multiple real zeros, since these also belong to h_0 and $\hat{h}_0$, and h_0 has only simple real zeros.

Theorem 9: Let h_1 and h_2 belong to $Z(\alpha,\beta)$. Then,

$$\operatorname{sgn}\{h_1(t)\} = \operatorname{sgn}\{h_2(t)\}, \quad -\infty < t < \infty$$

implies

$$h_1(t) = Ah_2(t), \quad -\infty < t < \infty$$

provided $\sigma(T)$, the number of sign changes of $h_1(t)$ in $(0;T)$, satisfies

$$(i) \quad \limsup_{T \to \infty} \frac{\sigma(T)}{T} > \frac{\beta - \alpha}{\pi}.$$

Furthermore, (i) is always satisfied if $h_1 \not\equiv 0$ and (ii) $\alpha > \beta/2$.

Proof: We may assume that $h_1 \not\equiv 0$ and, hence, that $h_2 \not\equiv 0$. Otherwise the conclusion is trivially true. Then, since h_1 and h_2 vanish together on a set of uniqueness for $B_\infty(\lambda)$, we have from Theorem 5,

$$\frac{h_1(t)}{\hat{h}_1(t)} \equiv \frac{h_2(t)}{\hat{h}_2(t)} \equiv M(t).$$

The poles and zeros of $M(t)$ are determined by any non-null function in $B_\infty(\alpha,\beta)$ that vanishes at the points of sign change. The zeros of $M(t)$ identify the zeros of h_i that are not common to $\hat{h}_i (i = 1,2)$. That is, the free zeros of h_i are missing. All we have to show is that the locations of the free zeros of, say h_1, are uniquely determined by the zeros of M and the points of sign change of h_1 and h_2. It would then follow that h_1 and h_2 have the same set of zeros, and then the conclusion follows from (42).

Denote by $\{\xi_k\}$ the free (real, simple) zeros of h_1 and by $\{\tau_k\}$ the zeros of M and define

$$\Pi_1(t) = \prod_k \left(1 - \frac{t}{\xi_k}\right) e^{t/\xi_k} \tag{46}$$

$$\Pi_0(t) = \prod_k \left(1 - \frac{t}{\tau_k}\right) e^{t/\tau_k}, \tag{47}$$

where we have assumed, as a matter of convenience in writing, that $\xi_k \neq 0$, $\tau_k \neq 0$. [When the zeros of h_1 are thus separated into two sets, the exponential factors are generally required to make the infinite products in (46) and (47) converge. We could have, for example, $\xi_k = k$, $k = 1,2,\cdots$.] We have by the Hadamard factorization Theorem*

$$h_1(t) = h_1(0)e^{ct}\Pi_0(t)\Pi_1(t) \tag{48}$$

* See Ref. 2, page 22.

for some real c, which is irrelevant to the argument. The $\{\tau_k\}$ appear in conjugate pairs so both $\Pi_0(t)$ and $\Pi_1(t)$ are real-valued. We may assume that $h_1(0) > 0$. Then,

$$\text{sgn}\,\{h_1(t)\} = \text{sgn}\,\{\Pi_0(t)\}\,\text{sgn}\,\{\Pi_1(t)\} \quad -\infty < t < \infty. \tag{49}$$

Now $\Pi_0(t)$ and, hence, sgn $\{\Pi_0(t)\}$ are (in principle) given and sgn $\{h_1(t)\}$ is known [except at even-order zeros of $h_1(t)$]. We have

$$\text{sgn}\,\{\Pi_1(t)\} = \text{sgn}\,\{\Pi_0(t)\}\,\text{sgn}\,\{h_1(t)\} \quad \text{for almost all } t, \tag{50}$$

and since the zeros of $\Pi_1(t)$ are real and simple, (50) defines them uniquely; i.e., they are the points where the function on the right *changes* sign.

VI. THE ZEROS OF FULL-CARRIER LOWER-SIDEBAND SIGNALS

In connection with condition (28), which is a sufficient condition for a function to have no free zeros and, hence, to belong to $Z(\alpha,\beta)$, it is worth noting that the condition is also a sufficient condition for the function to have only real simple zeros. In particular, functions of the form

$$h(t) = \text{Re}\,[\{1 + x(t) - i\hat{x}(t)\}e^{i\beta t}], \tag{51}$$

where

$$|x(t)| < 1 \tag{52}$$

and

$$x,\hat{x} \text{ (real) belong to } B_\infty(\lambda), \quad 0 \leqq \lambda < \beta, \tag{53}$$

which are called "full-carrier" lower-sideband signals, have only real simple zeros. We have the following more general result.

Theorem 10: Let f be a (non-null) bounded band-limited function whose spectrum is confined to the interval $[-\lambda,\lambda]$, $0 \leqq \lambda < \infty$, but to no smaller interval; i.e., $e^{i\mu t}f(t)$ does not belong to $B_\infty(\lambda)$ for any μ different from zero. Also, let $f(\tau)$, $\tau = t + iu$, be zero free in the closed lower half-plane $u \leqq 0$. Then the zeros of the function h defined by

$$h(t;\mu) = \text{Re}\,\{f(t)e^{i\mu t}\} \tag{54}$$

are real and simple provided $\mu > 0$, or provided $\mu \geqq 0$ if $f \not\equiv$ constant.

Note that $h(t;\mu)$ in (54) need not be bandpass; i.e., we do not require $\mu > \lambda$. The function is just a special kind of band-limited function. The result is independent of λ so long as $\lambda < \infty$. The only significance of λ is to indicate that the spectral end points are centered about the origin.

Proof: The conclusion is trivial for f = constant, so we assume that $f \neq$ constant and, consequently, has an infinite number of zeros. From (42) we have

$$f(t) = f(0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{\tau_k}\right), \tag{55}$$

where the product converges conditionally with the provision $|\tau_{k+1}| \geq |\tau_k|$. We have by hypothesis

$$\tau_k = a_k + ib_k, \quad \text{where} \quad b_k > 0. \tag{56}$$

Then,

$$\begin{aligned} 2h(t) &= e^{i\mu t} f(t) + e^{-i\mu t} \bar{f}(t) \\ &= e^{i\mu t} f(t) \left\{1 + \frac{\bar{f}(t)}{f(t)} e^{-i2\mu t}\right\}. \end{aligned} \tag{57}$$

We have

$$B(t) \equiv \frac{\bar{f}(t)}{f(t)} = \frac{\bar{f}(0)}{f(0)} \frac{\prod_{k=1}^{\infty} \left(1 - \frac{t}{\bar{\tau}_k}\right)}{\prod_{k=1}^{\infty} \left(1 - \frac{t}{\tau_k}\right)} = \frac{\bar{f}(0)}{f(0)} \prod_{k=1}^{\infty} \frac{\left(1 - \frac{t}{\bar{\tau}_k}\right)}{\left(1 - \frac{t}{\tau_k}\right)}, \tag{58}$$

where the last product converges absolutely since

$$\sum \left(\frac{1}{\bar{\tau}_k} - \frac{1}{\tau_k}\right) = 2 \sum \frac{\mathrm{Im}\{\tau_k\}}{|\tau_k|^2}$$

converges absolutely.* Since

$$\left| \frac{1 - \dfrac{t + iu}{a_k - ib_k}}{1 - \dfrac{t + iu}{a_k + ib_k}} \right|^2 = \frac{(a_k - t)^2 + (u + b_k)^2}{(a_k - t)^2 + (u - b_k)^2} < 1 \quad \text{for } u < 0, \tag{59}$$

we have

$$|B(t + iu)| < 1 \quad \text{for } u < 0. \tag{60}$$

Hence,

$$|B(t + iu) e^{-i2\mu(t+iu)}| = e^{2\mu u} |B(t + iu)| < 1 \quad \text{for } u < 0. \tag{61}$$

Therefore, since $f(\tau)$ does not vanish on the real axis nor in the lower half-plane, it follows from (57) and (61) that $h(\tau)$ has no complex zeros.

* See Ref. 2, Theorem 6.3.14, page 86.

Now $\log f(\tau)$ is analytic in the upper half-plane and we may write

$$f(t) = |f(t)|e^{i\varphi(t)}, \tag{62}$$

where

$$\varphi(t) = \text{Im}\,\{\log f(t)\} \quad \text{and (say)} \tag{63}$$

$$0 \leqq \varphi(0) < 2\pi. \tag{64}$$

Then,

$$h(t) = |f(t)| \cos\,\{\varphi(t) + \mu t\}. \tag{65}$$

Since $f(t)$ does not vanish on the real line, the zeros of $h(t)$ are the zeros of $\cos\,\{\varphi(t) + \mu t\}$. Now if $\cos\,\{\varphi(t_k) + \mu t_k\} = 0$, then

$$c'(t_k) \equiv \frac{d}{dt} \cos\,(\varphi(t) + \mu t)\big|_{t=t_k} = -\{\mu + \varphi'(t_k)\} \sin\,\{\varphi(t_k) + \mu t_k\} \tag{66}$$

or

$$|c'(t_k)| = |\mu + \varphi'(t_k)|. \tag{67}$$

We have

$$B(t) = e^{-2i\varphi(t)} \tag{68}$$

and

$$\frac{B'(t)}{B(t)} = -\,2i\varphi'(t) = \sum \left\{\frac{1}{t - \bar{\tau}_k} - \frac{1}{t - \tau_k}\right\} = -2i\sum \frac{b_k}{|t - \tau_k|^2} \tag{69}$$

or

$$\varphi'(t) = \sum \frac{b_k}{|t - \tau_k|^2} > 0. \tag{70}$$

Hence, if $\mu \geqq 0$,

$$|c'(t_k)| > 0 \tag{71}$$

and, therefore, all zeros of h are real and simple. Since $\text{Im}\,\{f(t)e^{i\mu t}\} = \text{Re}\,\{-if(t)e^{i\mu t}\}$ the conclusion of Theorem 10 also holds for $\text{Im}\,\{f(t)e^{i\mu t}\}$ and, since $\varphi'(t) > 0$ and since the zeros of $\cos\,\{\mu t + \varphi(t)\}$ and $\sin\,\{\mu t + \varphi(t)\}$ interlace ($\mu > 0$), we have proved

Theorem 11: If f satisfies the hypotheses of Theorem 10, then the functions

$$h_1(t;\mu) = \text{Re}\,\{f(t)e^{i\mu t}\}$$

$$h_2(t;\mu) = \text{Im}\,\{f(t)e^{i\mu t}\}$$

have all real, simple, interlacing zeros for $\mu > 0$.

In applying Theorems 10 and 11 to functions of the form (51), we set

$$g(t) = 1 + x(t) - i\hat{x}(t).$$

The spectrum of g is confined to an interval $[-\alpha,0]$ (where $\alpha \leqq \lambda$) but not to an interval $[-\alpha,\epsilon]$, where $\epsilon < 0$. Otherwise $\{1 + x(t)\} > 0$ would belong to $B_\infty(\epsilon,\alpha)$ and would, therefore, have an infinite number of sign changes, which is a contradiction. We suppose further that the spectrum is not confined to a smaller interval; i.e., that α is the left end point of the spectrum. We then set

$$f(t) = g(t) \exp\left\{\frac{i\alpha t}{2}\right\},$$

so that f meets the hypotheses of the theorem. Then writing

$$\begin{aligned} h &= \text{Re}\,\{g(t) \exp(i\beta t)\} \\ &= \text{Re}\left\{f(t) \exp\left(i\beta t - \frac{i\alpha t}{2}\right)\right\} \end{aligned}$$

we may state the result as

Corollary 10.1: A function of the form (51) has only real simple zeros when the condition in (53) is replaced by $\beta > \lambda/2 \geqq 0$.

When $\beta > \lambda$, as in (53), h has a Hilbert transform $\hat{h}(t) = \text{Im}\,[\{1 + x(t) - i\hat{x}(t)\}e^{i\beta t}]$. So we have

Corollary 11.1: A function h of the form (51) and its Hilbert transform $\hat{h}$ have only real, simple, interlacing zeros.

We state one more result which follows from the proof of Theorem 10:

Theorem 12: Let f be a bounded (non-null) band-limited function whose spectrum is confined to the interval $[\alpha,\beta]$ but to no smaller interval, and let $f(\tau)$, $\tau = t + iu$, be zero-free in the upper half-plane $u \geqq 0$. Then the phase function $\varphi(t)$, defined uniquely by

(i) $\varphi(t)$ *is continuous*

(ii) $0 \leqq \varphi(0) < 2\pi$

(iii) $f(t) = |f(t)|e^{i\varphi(t)}$

satisfies

(iv) $\varphi'(t) \leqq \dfrac{\alpha + \beta}{2}.$

It follows, in particular, if $x(t)$ is a positive function in $B_\infty(\lambda)$ and has a Hilbert transform $\hat{x}(t)$ and

$$\varphi(t) = \tan^{-1} \frac{\hat{x}(t)}{x(t)},$$

$$-\frac{\pi}{2} < \varphi(t) < \frac{\pi}{2},$$

then

$$\varphi'(t) \leqq \lambda/2.$$

(Since x is positive, the smallest interval containing the spectrum of $x + i\hat{x}$ is $[0,\lambda']$ for some $\lambda' \leqq \lambda$.) We note that without further qualification, x must be bounded away from zero in order to obtain a (finite) lower bound for φ', as the example

$$x(t) = 1 + a \cos t, \quad (a = 1 - \epsilon)$$

shows.

VII. DISCUSSION AND CONCLUSIONS

The zeros of a bandpass function h that can be moved around without destroying the bandpass property of h; i.e., the free zeros of h play a key role in the problem here and it is safe to assume that they will be important in other problems. We have shown (Theorem 1) that the free zeros of h are simply the common zeros of h and its Hilbert transform $\hat{h}$ (whether or not $h(t)$ is real). It follows (Theorem 2) that moving a free zero of h simply alters its Hilbert transform in the same way; i.e., only the corresponding (common) zero of $\hat{h}$ is moved.

If we are given a large enough subset S of the zeros of h, then (Theorem 5) S determines $h/\hat{h}$. Without further qualification of S or h, this is all that S determines. If real-valued h has enough sign changes, slightly more (roughly speaking) than $\cos \lambda t$, where λ is the width of the passband (of the whole class), then the zero crossings $\{t_k\}$ constitute a set S which determines $h/\hat{h}$. This, without further qualification, is all the information the zero crossings may convey. If, in addition, it is known that h has no free zeros, then under the stipulated conditions $\{t_k\}$ determines h within a constant multiplier.

If h has free zeros, then we cannot determine (a multiple of) $h(t)$ from sgn $\{h(t)\}$, because (Theorem 4) there are other functions in the same class having the same signum function. In this connection, we note the following:

In the representation

$$h(t;\mu) = \text{Re}\,\{f(t)e^{i\mu t}\}, \quad \mu > \lambda/2,$$

where f is regarded as a fixed (complex-valued) function in $B_\infty(\lambda/2)$ and μ as a parameter, the zero crossings of $h(t;\mu)$ for arbitrarily large μ give no more information about f than for $\mu = 3\lambda/2 + \epsilon$, $\epsilon > 0$ (when the band spans less than an octave). The free zeros of $h(t;\mu)$, which are crucial to identifying h (or f) are invariant with μ.

These results may be generalized and specialized in various ways. We should note a specialization to functions of the form

$$h(t) = \cos \mu t - q(t) \sin \mu t,$$

where $q(t)$, real, belongs to $B_\infty(\lambda/2)$ and $\mu > \lambda/2$. This describes a common sort of phase modulation. Here $h(t)$ has no free zeros because the corresponding function in (13),

$$f(t) = 1 + iq(t),$$

clearly has no real zeros, and if ξ is a complex zero of f we have

$$q(\xi) = i$$

and, hence, since $q(t)$ is real,

$$q(\bar{\xi}) = -i$$

and so

$$f(\bar{\xi}) = 2.$$

Thus (Corollary 3.1) h has no free zeros. Then, if we consider two functions h_1 and h_2 of this form and return to the basic identity (37), we have $p_1 = p_2 = 1$ and

$$\hat{h}_1(t)h_2(t) - \hat{h}_2(t)h_1(t) = q_1(t) - q_2(t),$$

which belongs to $B_\infty(\lambda/2)$ rather than $B_\infty(\lambda)$. Now,

$$h_i(k\pi/\mu) = \cos k\pi = (-1)^k, \quad k = 0, \pm 1, \pm 2, \cdots, \quad (i = 1,2)$$

so $h_i(t)$ has at least as many sign changes as $\cos \mu t$. Thus, if $\mu > \lambda/2$ (just enough for high-pass), the zero crossings $\{t_k\}$ of h_i constitute a set of uniqueness for $B_\infty(\lambda/2)$, which is all we need to conclude that

$$\operatorname{sgn} h_1(t) \equiv \operatorname{sgn} h_2(t)$$

implies

$$q_1(t) - q_2(t) \equiv 0$$

i.e.,

$$h_1(t) \equiv h_2(t).$$

In this case the recovery problem is much simpler than in the general case. Here we are given

$$q(t_k) = \cot \mu t_k, \quad \text{all } t_k \text{ in } S, \quad q \text{ in } B_\infty(\lambda/2),$$

and seek $q(t)$; i.e., knowing $p(t) = 1$ vastly simplifies the problem.

As to generalizations, the results may be extended to bandpass functions that are not bounded (e.g., sample functions of gaussian processes). We can replace $B_\infty(\lambda)$ by $B(\lambda)$, which consists of restrictions to the real line of entire functions of exponential type λ whose growth (on the real line) is less than exponential (see Ref. 7). The zeros of these entire functions have ordinary densities, separately in the right and left half-planes, which are equal and do not exceed λ/π.[7] Hence, sets $\{t_k\}$ of upper density greater than λ/π constitute sets of uniqueness for $B(\lambda)$.

It is clear from the Hadamard factorization

$$f(t) = f(0)e^{ct} \prod_{k=1}^{\infty} \left(1 - \frac{t}{\tau_k}\right) e^{t/\tau_k}$$

that the zeros $\{\tau_k\}$ of real-valued f in $B(\lambda)$ determine f within a constant multiplier. Since the τ_k occur in conjugate pairs, the product is real-valued on the real axis and, hence, the exponent c must be real. Then c will be determined by the condition that the growth on the real axis be less than exponential.

Then we define $B(\alpha,\beta)$ analogous to $B_\infty(\alpha,\beta)$, and for h in $B(\alpha,\beta)$, we let $\hat{h}$ be defined by the right-hand side of (6) with f_1 and f_2 in $B(\lambda/2)$, and simply call it the generalized Hilbert transform of h. It is not important what we call it; the free zeros of h are still the common zeros of h and $\hat{h}$, or equivalently the common zeros of p and q. Then Theorem 7 must be generalized to $B(\alpha,\beta)$. It is clear that the proof in Ref. 9 extends easily, so all the uniqueness results may be extended to $B(\alpha,\beta)$.

In connection with this generalization, it might be interesting to study the free zeros of sample functions of bandpass gaussian processes $\{h\}$. The free zeros are going to be very rare (in the ergodic case) to say the least. It may be advisable to begin the study with the case of periodic sample functions.

There are still other questions that arise in connection with the problem considered here. For example, we have not shown that given an arbitrary real h in $B_\infty(\alpha,\beta)$ there is a corresponding function in $Z(\alpha,\beta)$ having the same signum function. The difficulty occurs when h has an infinite number of free zeros which, for example, may be complex and restricted to the right half-plane and have positive density there. (Such functions can be constructed on the basis of Corollary 3.1.) The Hadamard product composed of the free zeros will then not even belong to the broader class $B(\cdot)$ just discussed. The remaining zeros will not have equal densities in the right and left half-planes and, hence, the Hadamard product composed of these zeros will not belong to $B(\cdot)$. There seems to be no way to replace the free zeros with non-free zeros and obtain a function in $Z(\alpha,\beta)$ with the same signum function as h. It appears that the argument can be completed to prove that the proposition is false.

Another problem is that of characterizing those h for which there is not another distinct function in the *whole* class $B_\infty(\alpha,\beta)$ having the same signum function. Of course, h must belong to $Z(\alpha,\beta)$ but now other arguments of a Fourier nature are required. The end points of the spectrum play an important role in this problem. For example, cos αt and cos βt are special functions in $Z(\alpha,\beta)$ which for $\beta < 2\alpha$ meet the conditions of the problem, a result we state without proof. It appears that the "full-carrier" sideband signals, which have spectrum at one or the other end points, are also special functions of this type when $\beta < 2\alpha$. The decay of $h(t)$ also enters in the problem; i.e., $(1 + t^2)h(t)$ must not belong to $B_\infty(\alpha,\beta)$. The basic idea is that one should not be able to multiply $h(t)$ by a positive function and obtain a function in $B_\infty(\alpha,\beta)$. This obviously will be possible if the spectrum of h is confined to $[\alpha',\beta']$ (and $[-\beta',-\alpha']$), where $\alpha < \alpha' < \beta' < \beta$.

Given $h(t)$ in a form other than (13) with an explicit factorization of f, it is obviously difficult to determine whether or not h belongs to $Z(\alpha,\beta)$. However, it is easy to synthesize functions in $Z(\alpha,\beta)$, e.g., the full-carrier sideband signals.

The problem of actually recovering functions in $Z(\alpha,\beta)$ from their zero crossings appears to be difficult (to say the least) under the most general conditions for uniqueness. A general "method" suggested by the proof of Theorem 9 requires first finding any non-null test function in $B_\infty(\alpha,\beta)$ that merely vanishes at the points of sign change or some subset of the points that constitute a set of uniqueness for $B_\infty(\lambda)$. However, this in itself is a difficult, if not intractable, problem except in the simple periodic case. Assuming such a test function to be found, it will, in general, have complex free zeros and/or real free zeros which the sought after function h does not have. So, in effect, the test function and its Hilbert transform must be factored to discard common zeros (free zeros), which amounts to finding the zeros of $M(t)$ in (38), or the poles and zeros of $N(t)$ in (40). Then one constructs as in the proof of Theorem 9, a function $\Pi_0(t)$ with the zeros of $M(t)$, i.e., the non-free zeros of h. Then the missing (real simple) free zeros of h can be determined by comparing the sign changes of $\Pi_0(t)$ and the given sign changes of $h(t)$ as in (50).

The overall recovery procedure is obviously hopeless except in the case of periodic functions. There may be some simpler procedure under more restrictive hypotheses; e.g., condition (27), ensuring that f be zero-free in the closed upper half-plane. Condition (28), ensuring that f be zero-free in the closed lower half plane, was shown to imply that the corresponding h (e.g., a full-carrier lower-sideband signal) has all real simple zeros, in which case h can be recovered by forming an infinite product having simple zeros at the points of sign change. This fact, aside from questions of practicality, might suggest a preference in full-carrier sideband transmission for the lower sideband.

The results here have theoretical interest in that they provide a satisfactory answer to the general question as to what information (in our sense) is conveyed by the zero crossings of bandpass functions. As far as practicality is concerned, the results cannot be extrapolated with abandon to "almost bandpass" functions. Although there is no argument with the assertion that practical signals can be closely approximated with bandpass signals, it does not follow that there even exists a bandpass signal (to which the results apply) with the same zero crossings as the practical signal, much less one which has the same zero crossings and is everywhere close to the practical signal. Clearly one must have a very severely constrained class of signals in order to assert that the zero crossings "closely" determine the signals.

APPENDIX

Here we sketch a proof of the fact

$$\lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)f(t)e^{-i\omega t}dt=0,\quad \text{for } |\omega|>\lambda,\qquad f \text{ in } B_\infty(\lambda).\tag{72}$$

First we set

$$h(t)=h(t;\omega)=f(t)e^{-i\omega t},\quad f \text{ in } B_\infty(\lambda)\tag{73}$$

and observe that for $|\omega|-\lambda=\alpha>0$ (ω = real), h belongs to the class $H_\infty(\alpha)$ consisting of all bounded functions h (high-pass functions) satisfying

$$\int_{-\infty}^{\infty}g(t)h(t)dt=0\quad \text{all } g \text{ in } B_1(\alpha).\tag{74}$$

Indeed, $f(t)g(t)$ belongs to $B_1(\alpha+\lambda)$ and, hence, its Fourier transform is continuous and vanishes outside $(-\beta,\beta)$, $\beta=\alpha+\lambda$.

Let us then define

$$C(t;T)=1-\frac{|t|}{T},\quad |t|\leqq T \qquad\qquad = 0,\quad |t|>T.\tag{75}$$

Then we wish to prove

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}C(t;T)h(t)dt=0,\quad h \text{ in } H_\infty(\alpha)\quad(\alpha>0).\tag{76}$$

There are several ways to prove this. In Ref. 4 we used the notion of the "unbiased" integral of $h(t)$, denoted by $h^{(-1)}(t)$, which is a particular integral of h also belonging to $H_\infty(\alpha)$. In general, we may define the nth unbiased integral of h by

$$h^{(-n)}(t) = \int_{-\infty}^{\infty} h(x)K_n(t-x)dx, \quad n = 1, 2, \cdots, \qquad h \text{ in } H_\infty(\alpha) \tag{77}$$

where K_n is any kernel of L_1 whose Fourier transform satisfies

$$\int_{-\infty}^{\infty} K_n(t)e^{-i\omega t}dt = (i\omega)^{-n} \quad \text{for } |\omega| \geqq \alpha. \tag{78}$$

Then we can show that $h^{(-n)}$ in fact does satisfy

$$\int_a^b h^{(-n)}(t)dt = h^{(-n-1)}(b) - h^{(-n-1)}(a) \qquad (-\infty < a < b < \infty). \tag{79}$$

It suffices to show this for $n = 0$, $h^{(0)} \equiv h$, and then use induction.

Achieser[10] shows (in another context) that the minimal L_1-norm kernels have norm

$$\|K_n\|_1 = \alpha^{-n}\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(n+1)}}{(2k+1)^{n+1}} = \alpha^{-n}M_n. \tag{80}$$

Then we have, integrating twice by parts,

$$\int_{-T}^{T} C(t;T)h(t)dt = \frac{1}{T}\{-2h^{(-2)}(0) + h^{(-2)}(T) + h^{(-2)}(-T)\} \tag{81}$$

and, hence,

$$\left|\int_{-T}^{T} C(t;T)h(t)dt\right| \leqq \frac{4M_2}{\alpha^2 T}\sup_t |h(t)|, \quad h \text{ in } H_\infty(\alpha). \tag{82}$$

Then (76) and (72) follow from (82).

Actually, for h of the form (73) we can replace M_2 in (82) by 1. For h having one-sided spectrum, say the half line $[\alpha,\infty)$, $\alpha > 0$, we only require

$$\int_{-\infty}^{\infty} K_n(t)e^{-i\omega t}dt = (i\omega)^{-n} \quad \text{for } \omega \geqq \alpha > 0. \tag{83}$$

Here we may obtain the minimal-norm kernels simply by making their

Fourier transforms even about $\omega = \alpha$. It then follows from convexity that

$$K_n(t) = (i)^{-n} p_n(t) e^{i\alpha t}, \quad \text{where } p_n(t) > 0 \tag{84}$$

and

$$\int_{-\infty}^{\infty} |K_n(t)| dt = \int_{-\infty}^{\infty} p_n(t) dt = \alpha^{-n}. \tag{85}$$

In general, if one defines a class of bounded functions having a spectral gap (a,b) by an orthogonality condition similar to (74), then a simple modification of the proof gives the gratifying result that their Fourier integrals are actually summable $(C,1)$ to zero in the gap.

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